1 Heating an Office Building (Newton's Law of Cooling)

Background Suppose that in Winter the daytime temperature in a certain office building is maintained at $70°F$. The heating is shut off at 10 P.M. and turned on again at 6 A.M. On a certain day the temperature inside the building at 2 A.M was found to be $65°F$. The outside temperature was $50°F$ at 10 P.M. and had dropped to $40°F$ by 6 A.M.

Problem What was the temperature inside the building when the heat was turned on at 6 A.M.

Physical Information Experiments show that the time rate of change of the temperature T of a body B (which conducts heat well, as, for example, a copper ball does) is proportional to the difference between T and the temperature of the surrounding medium (Newton's law of cooling).

Solution Step 1: Setting up a model Let $T(t)$ be the temperature inside the building and T_A the outside temperature (assumed to be constant in Newton's Law). Then by Newton's law,

$$\frac{dT}{dt} = k(T - T_A) \tag{1}$$

Such experimental laws are derived under idealized assumptions that rarely hold exactly. However, even if a model seems to fit the reality only poorly (as in the present case), it will still give valuable qualitative information. To see how good a model is, the engineer will collect experimental data and compare them with the calculations from the model.

Solution Step 2: General Solution We cannot solve (1) because we do not know T_A, just that it varied between $50°F$ and $40°F$, so we follow the **Golden Rule**: *If you cannot solve your problem, try to solve a simpler one.* We solve (1) with the unknown function T_A replaced by the average of the two known values, or $45°F$. For physical reasons we may expect that this will give us a reasonable approximate value of T at 6 A.M.

For constant $T_A = 45$ (or any other *constant* value) the ODE (1) is separable. Separation, integration, and taking exponents gives the general solution

$$\frac{dT}{T-45} = k\,dt, \qquad \ln|T-45| = kt + c^*, \qquad T(t) = 45 + ce^{kt} \qquad (c = e^{c^*})$$

Solution Step 3: Particular Solution We choose 10 P.M. to be $t = 0$. Then the given initial condition is $T(0) = 70$ and yields a particular solution, call it T_p. By substitution

$$T(0) = 45 + ce^0 = 70, \qquad c = 70 - 45 = 25, \qquad T_p(t) = 45 + 25e^{kt}.$$

Solution Step 4: Determination of k We use $T(4) = 65$, where $t = 4$ is 2 A.M. Solving algebraically for k and inserting k into $T_p(t)$ gives (see Figure **??**)

$$T_p(4) = 45+25e^{4k} = 65, \qquad e^{4k} = 0.8, \qquad k = \frac{1}{4}\ln 0.8 = -0.056, \qquad T_P(t) = 45+25e^{-0.056t}.$$

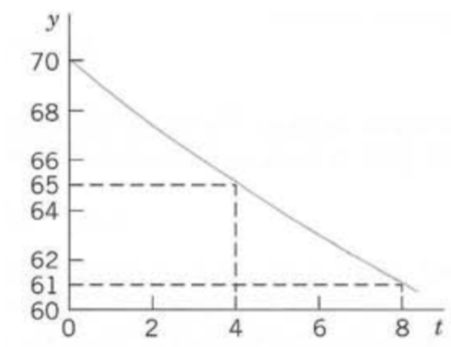

Figure 1: Particular Solution (temperature)

Solution Step 4: Answer and Interpretation 6 A.M. is $t = 8$ (namely 8 hours after 10 P.M.), and

$$T_p(8) = 45 + 25e^{-0.056\cdot 8} = 61.$$

Hence the temperature in the building dropped by $9°F$ from $70°F$ to $61°F$, a result that looks reasonable.