1 Mixing Problem

Background Mixing problems occur quite frequently in chemical industry. We explain here how to solve the basic model involving a single tank (see the figure on the right). The tank contains 1000 gal of water in which initially 100 lb of salt is dissolved. Brine – i.e. salt water – runs in at a rate of 10 gal/min, and each gallon contains 5 lb of dissolved salt. The mixture in the tank is kept uniform by stirring. Brine runs out at 10 gal/min.

Problem Find the amount of salt in the tank at any time t

Solution Step 1: Setting up a model Let $y(t)$ denote the amount of salt in the tank at time t. Its time rate of change is

$$y' = \text{salt inflow rate} - \text{salt outflow rate} \tag{1}$$

5 lb times 10 galgives an inflow of 50 lb of salt. Now, the outflow is 10 gal of brine. This is 10/1000 = 0.01 (=1%) of the total brine content in the tank, hence 0.01 of the salt content $y(t)$, that is, $0.01y(t)$. Thus, from (1) we obtain the following ODE as a model:

$$y' = 50 - 0.01y = -0.01(y - 5000). \tag{2}$$

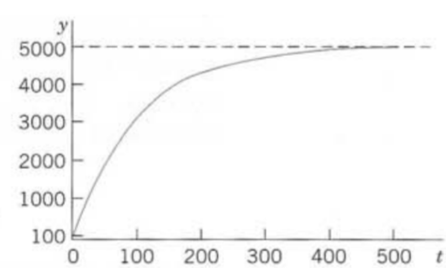
Solution Step 2: Solution of the Model The ODE (2) is separable. Separation, integration, and taking exponents on both sides gives

$$\frac{dy}{y-5000} = -0.01dt, \qquad \ln|y - 5000| = -0.01t + c^*, \qquad y - 5000 = ce^{-0.01t}.$$

Initially, the tank contains 100 lb of salt. Hence $y(0) = 100$ is the initial condition that will give the unique solution. Substituting $y = 100$ and $t = 0$ in the last equation gives $100 - 5000 = ce^0 = c$. Hence $c = 4900$. Hence the amount of salt in the tank at time t is

$$y(t) = 5000 - 4900e^{-0.01t}. \tag{3}$$

This function (see the graph on the right) shows an exponential approach to the limit 5000 lb. Can you explain physically that $y(t)$ should increase with time? That its limit is 5000 lb? Can you see the limit directly from the ODE?



Salt content $y(t)$