1 Heating an Office Building (Newton’s Law of Cooling)

Background

Suppose that in Winter the daytime temperature in a certain office building is maintained at $70^\circ F$. The heating is shut off at 10 P.M. and turned on again at 6 A.M. On a certain day the temperature inside the building at 2 A.M was found to be $65^\circ F$. The outside temperature was $50^\circ F$ at 10 P.M. and had dropped to $40^\circ F$ by 6 A.M.

Problem

What was the temperature inside the building when the heat was turned on at 6 A.M.

Physical Information

Experiments show that the time rate of change of the temperature T of a body B (which conducts heat well, as, for example, a copper ball does) is proportional to the difference between T and the temperature of the surrounding medium (Newton’s law of cooling).

Solution Step 1: Setting up a model

Let $T(t)$ be the temperature inside the building and T_A the outside temperature (assumed to be constant in Newton's Law). Then by Newton's law,

$$\frac{dT}{dt} = k(T - T_A) \quad (1)$$

Such experimental laws are derived under idealized assumptions that rarely hold exactly. However, even if a model seems to fit the reality only poorly (as in the present case), it will still give valuable qualitative information. To see how good a model is, the engineer will collect experimental data and compare them with the calculations from the model.

Solution Step 2: General Solution

We cannot solve (1) because we do not know T_A, just that it varied between $50^\circ F$ and $40^\circ F$, so we follow the Golden Rule: *If you cannot solve your problem, try to solve a simpler one.*We solve (1) with the unknown function T_A replaced by the average of the two known values, or $45^\circ F$. For physical reasons we may expect that this will give us a reasonable approximate value of T at 6 A.M.

For constant $T_A = 45$ (or any other *constant*value) the ODE (1) is separable. Separation, integration, and taking exponents gives the general solution

$$\frac{dT}{T - 45} = k\,dt, \quad \ln|T - 45| = kt + c^*, \quad T(t) = 45 + ce^{kt} \qquad (c = e^{c^*})$$

Solution Step 3: Particular Solution

We choose 10 P.M. to be $t = 0$. Then the given initial condition is $T(0) = 70$ and yields a particular solution, call it T_p. By substitution

$$T(0) = 45 + ce^0 = 70, \quad c = 70 - 45 = 25, \quad T_p(t) = 45 + 25e^{kt}$$

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Solution Step 4: Determination of k

We use $T(4) = 65$, where $t = 4$ is 2 A.M. Solving algebraically for k and inserting k into $T_p(t)$ gives (see Figure LABEL:fig:temp)

$$T_p(4) = 45 + 25e^{4k} = 65, \quad e^{4k} = 0.8, \quad k = \frac{1}{4}\ln 0.8 = -0.056, \quad T_P(t) = 45 + 25e^{-0.056t}$$

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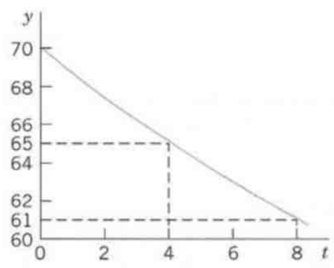

Figure 1: Particular Solution (temperature)

Solution Step 4: Answer and Interpretation

6 A.M. is $t = 8$ (namely 8 hours after 10 P.M.), and

$$T_p(8) = 45 + 25e^{-0.056 \cdot 8} = 61$$

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Hence the temperature in the building dropped by $9^{\circ}F$ from $70^{\circ}F$ to $61^{\circ}F$, a result that looks reasonable.