1 Maths

10pt

Problem 1.1 (Interval Intersections)
You are given a set of N open intervals $I_1, I_2, \ldots I_N$, with the property that:

$$\forall i, j. I_i \cap I_j \neq \emptyset$$

Prove **by induction** that:

$$\forall N \geq 2. I_1 \cap I_2 \cap \ldots I_N \neq \emptyset$$

2 Abstract Data Types

10pt

Problem 2.1 (ADT for UNN and prime numbers)
Design an ADT for unary natural numbers. Write a procedure that checks whether a number is prime.

3 Standard ML

20pt

Problem 3.1 (Game)
Four players A,B,C,D are playing the following game: They have a number of red and green stones and one blue stone arranged in a circle. (We will represent the circle by a list). The players perform the following actions in turn:
Player A replaces the first red stone after the blue stone by a green stone.

```
For example: [#``r'',#``r'',#``b'',#``g'',#``r'',#``r'']
would become [#``r'',#``r'',#``b'',#``g'',#``g'',#``r'']
```

Player B shifts the blue stone to the clockwise (to the right) by 3 replacing all the red sones he finds by green stones, if he reaches the end of the "list" he starts at the beginning:

```
For example: [#``r'',#``r'',#``b'',#``g'',#``r'']
would become [#``b'',#``r'',#``g'',#``g'',#``g'']
```

Player C changes the stone after the blue stone to a green stone:

```
For example: [#``r'',#``r'',#``b'',#``g'',#``r'']
would become [#``r'',#``r'',#``b'',#``g'',#``r'']

             [#``r'',#``r'',#``b'',#``r'',#``r'']
would become [#``r'',#``r'',#``b'',#``g'',#``r'']
```

Player D shifts the blue stone to the left (counter clockweise) by 1, and puts a green stone in it's original place:

```
For example: [#``r'',#``r'',#``b'',#``g'',#``r'']
would become [#``r'',#``b'',#``g'',#``g'',#``r'']
```

The player who replaces the last red stone by a green stone wins.
Assuming player A starts first, and the players play in the order A,B,C,D, write a sml function that given the list with the arrangement of stones, determines which of the players will win, and how many moves player A makes. Don't forget to raise the appropriate exceptions.

Example and signature:

```
val game = fn : char list -> string * int
- game([#"r",#"g",#"r",#"b",#"r",#"g"]);
val it = ("A␣wins",2) : string * int
```

10pt

Problem 3.2 (Sum decomposition)
Design an SML function that takes an integer $n > 0$ and returns all the possible ways in which n can be written as sum of strictly positive integers. Encode the result as a string.

Function signature and example:

```
val decompose = fn : int -> string list
- decompose 3;
val it = ["3","2␣+␣1","1␣+␣2","1␣+␣1␣+␣1"] : string list
```

How many decompositions exist for an n ? (Write your answer and a short argument at the end of the source file)

4 Formal Languages

10pt

Problem 4.1 (Formal Languages)
You are given the alphabet $A = \{a, b, c\}$ and a $L := \bigcup_{i=0}^{\infty} L_i$, where $L_0 = \{a\}$ and $L_{i+1} = \{xxb, xcy \mid x, y \in \bigcup_{k=0}^{i} L_k\}$.

1. Determine the cardinality of L_2, **without** explicitly writing down the strings it contains.
2. For each of the strings below, determine whether it is in L. Explain why or why not!
   - $s_1 = accca$
   - $s_2 = acca$
   - $s_3 = acacaab$

7pt

Problem 4.2 (Code definitions)

Define the following concepts and give an example of each:

1. Character code.
2. String code.
3. Prefix code.

Why are prefix codes also string codes? 7pt

Problem 4.3 (Formal Languages and Concatenation and Intersection)
Given the alphabet $A = \{a, b\}$ and 3 formal languages in A $L_1 = \{a^{[n]} \mid n \in \mathbb{N}\}$, $L_2 = \{ba^{[n]} \mid n \in \mathbb{N}\}$, $L_3 = \{b^{[k]}a^{[2n]} \mid n \in \mathbb{N}, k \in \mathbb{N}\}$.

1. What is $L_1 \cap L_3$?
2. Write down three words that belong in $L_4 = \text{conc}(L_2, L_1)$.

5 Boolean Expressions

7pt

Problem 5.1 (Practising Quine McCluskey)
Use the algorithm of Quine-McCluskey to determine the minimal polynomial of the following function:

$x1$	$x2$	$x3$	f
F	F	F	F
F	F	T	F
F	T	F	T
F	T	T	F
T	F	F	T
T	F	T	T
T	T	F	F
T	T	T	T

7pt

Problem 5.2 (Model for Boolean Expressions)
Give a variable assignment φ for which all the following expressions evaluate to true.

1. $e_1 := x_1 * \overline{x_2} + \overline{x_2 + x_3} * \overline{x_1 + \overline{x_3}}$
2. $e_2 := \overline{x_1} * (x_2 * \overline{x_3}) + x_1 * (\overline{x_2} * x_3)$
3. $e_3 := (x_1 + x_2) * (x_2 + x_3)$

Show your reasoning using truth tables.

6 Propositional Logic

7pt

Problem 6.1 (Hilbert calculus)
Prove the following theorem of Hilbert Calculus (using Hilbert Calculus rules only!!! - and make sure you specify the rules used on the way)

$$(S \Rightarrow R) \Rightarrow S \Rightarrow S \Rightarrow R$$

10pt

Problem 6.2 (Natural deduction)

Prove the following theorem of Natural Deduction (using ND Calculus rules only!!! - give their short abbreviation too when applying them)

$$(P \Rightarrow Q) \Rightarrow (\neg P \vee Q)$$