Quizzes for General CS I (320101) Fall 2010

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For Course Purposes Only

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Contents

Quiz 2: Peano axioms and Induction(Given Sep. 20. 2010)

4pt

Problem 2.1 (Successor of three)
Prove or refute that $s(s(s(o)))$ is a unary natural number and that it has a successor that is also a unary natural number.

Note: Please use only the Peano Axioms for this proof.

Solution:
Proof: We will prove the statement using the Peano axioms:

P.1 o is a unary natural number (axiom P1)

P.2 $s(o)$ is a unary natural number (axiom P2 and 1.)

P.3 $s(s(o))$ is a unary natural number (axiom P2 and 2.)

P.4 $s(s(s(o)))$ is a unary natural number (axiom P2 and 3.)

P.5 $s(s(s(s(o))))$ exists and also is a unary natural number (axiom P2 and 4.)

□

8pt

Problem 2.2 (Geometric progression of base 2)
Prove or refute the law of the geometric progression:

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

Note: You do not have to go back to the Peano Axioms here, use ordinary mathematical language

Solution:
Proof: We prove the assertion by induction on n.

P.1.1 Base case: $n = 0$:

P.1.1.1 then $2^0 = 2^{0+1} - 1$, since $2^0 = 1$ by definition. □

P.1.2 $n \geq 1$:

P.1.2.1 For the step case let $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$.

P.1.2.2 We have

$$\sum_{i=0}^{n+1} 2^i = (\sum_{i=0}^{n} 2^i) + 2^{n+1} = 2^{n+1} - 1 + 2^{n+1} = 2^{n+1} \times 2 - 1 = 2^{n+2} - 1$$

P.1.2.3 This completes the step case. □

P.2 By proving the base case and the step case, the induction theorem ensures that the relation holds for every n. □

Quiz 3: Relations and Mathtalk(Given Sep. 27. 2010)

12pt

Problem 3.1 (Relation MathTalk)

Given a base set A and a relation $R \subseteq A \times A$, consider the following definitions:

1. $\forall a, b \in A. \langle a, b \rangle \in R \Rightarrow \langle b, a \rangle \in R \Leftrightarrow a = b$
2. $\forall a \in A. \exists^1 b \in S. \langle a, b \rangle \in R \Rightarrow (a \leq b)$
3. $\forall x, y, z \in A. (\langle z, y \rangle \in R \wedge \langle y, x \rangle \in R) \Rightarrow \langle z, x \rangle \in R$

For every one of the above definitions, translate from math talk to natural language, and if it is the case, state the meaning/concept of the respective definition.

Solution:

1. Antisymmetry. For all elements a and b in A if the tuple $\langle a, b \rangle$ is in the relation, then $\langle b, a \rangle$ can be in the relation only if $a = b$.
2. Random property. For all elements a and b in A if the tuple $\langle a, b \rangle$ is in the relation, then a is less than b.
3. Transitivity. For all elements a, b, c in A if $\langle z, y \rangle$ is in the relation and $\langle y, x \rangle$ is in the relation, then $\langle z, x \rangle$ must also be in the relation.

Quiz 4: Functions(Given Oct. 4. 2010)

12pt

Problem 4.1 (Function Properties)
Consider the following functions in $\mathbb{R} \to \mathbb{R}$

1. $f(x) = x^2 - x$
2. $g(x) = \frac{1}{5+e^x}$
3. $h(x) = x \cdot ln(3 + x)$

Your tasks are the following:

1. Are f, g, and h total? If not, determine the biggest set the function is defined on.
2. Are the three functions injective or not? Justify your answer.
3. Write down the expressions for:
   (a) $f \circ g$
   (b) $(h \circ f)^{-1}$
   (c) $g \circ g$

Solution:

1. f is not injective since $f(0) = f(1) = 0$. g is injective since it is a strictly decreasing function. The argument with $g(x_1) = g(x_2)$ resulting in $x_1 = x_2$ may also be used. h is not injective since $ln(3+x)$ and x are continuous strictly increasing functions. This means that for some $x \leq 0$ and some $0 \leq y$, $h(x) = h(y)$.
2. (a) $f \circ g(x) = f(\frac{1}{5+\exp x}) = \frac{1}{5+e^x}^2 - \frac{1}{5+e^x}$
   (b) $(h \circ f)^{-1} = f^{-1} \circ h^{-1}$ does not exist since f is not bijective (not injective from previous task).
   (c) $g \circ g = g(\frac{1}{5+\exp x}) = \frac{1}{5+e^{\frac{1}{5+e^x}}}$

Quiz 5: SML Datatypes(Given Oct. 10. 2010)

12pt

Problem 5.1 (Temperatures)

You are given the following SML datatype `temp` that represents temperatures in Fahrenheit and Celsius. You are asked to write a function `find` that returns the lowest temperature in a list.

```
datatype temp = Celsius of real | Fahrenheit of real;
fun find : temp list -> real;
```

Example:

```
- find([Celsius(12.0), Fahrenheit(52.0), Celsius(32.0)]);
val it = Fahrenheit 52.0 : temp
```

Note: The following formula holds for transforming Fahrenheit into Celsius:

$$t_C = (t_F - 32) \cdot \frac{5}{9}$$

Solution:

```
datatype temp = Celsius of real | Fahrenheit of real;

(* transform F into C to have a ‘‘common ground’’ *)
fun value(Celsius(x)) = x
  | value(Fahrenheit(x)) = (x-32.0)*5.0/9.0;

fun find([a]) = a
  | find(a::l) = let val x = find(l) in if value(a) > value(x) then x else a end;
```

Quiz 6: Abstract Data Types(Given Nov. 1. 2010)

6pt

Problem 6.1 (Abstract Data Type for Ground Terms)
Consider an abstract datatype with symbols $\mathbb{A}$ and $\mathbb{B}$ such that $[a\colon \mathbb{A}]$ and $[b\colon \mathbb{B}]$ are base declarations.

Extend this abstract data type so that the expressions $g(f(a,b,c),h(b,c))$ and $h(m(c),g(a,n(b)))$ are ground constructor terms and write it out formally.

Solution: A possible solution for the the ADT is:

$$\langle\{\mathbb{A},\mathbb{B}\},\{[a\colon \mathbb{A}],[b\colon \mathbb{B}],[c\colon \mathbb{A}],[f\colon \mathbb{A}\times\mathbb{B}\times\mathbb{A}\to\mathbb{A}],[g\colon \mathbb{A}\times\mathbb{A}\to\mathbb{A}],[h\colon \mathbb{B}\times\mathbb{A}\to\mathbb{A}],[m\colon \mathbb{A}\to\mathbb{B}],[n\colon \mathbb{B}\to\mathbb{A}]\}\rangle$$

Problem 6.2 (An abstract procedure) 6pt

Given the following ADT for lists of unary natural numbers

$$L := \langle \mathbb{L}, \mathbb{N}, [o\colon \mathbb{N}], [s\colon \mathbb{N} \to \mathbb{N}], [nil\colon \mathbb{L}], [cons\colon \mathbb{N} \times \mathbb{L} \to \mathbb{L}] \rangle$$

and an abstract procedure for appending two lists,

$$\langle @{::}\mathbb{L} \times \mathbb{L} \to \mathbb{L}\,;\, \{@(cons(n, L_1), L_2) \rightsquigarrow cons(n, @(L_1, L_2)), @(nil, L_2) \rightsquigarrow L_2\} \rangle$$

Provide an abstract procedure for reversing lists!

Solution: Please refer to slide 90, page 58 of the lecture notes. The definition of reverse is:

$$\langle \rho{::}\mathbb{L} \to \mathbb{L}\,;\, \{\rho(cons(a, L)) \rightsquigarrow @(\rho(L), cons(a, nil)), \rho(nil) \rightsquigarrow nil\} \rangle$$

Quiz 7: Abstract data types(Given Nov. 8. 2010)

12pt

Problem 7.1 (Surprise Functions)

You are given the following SML code:

```
exception RandomException;
fun f(0, n) = f(0,n)
   | f(1, _) = raise RandomException
   | f(2, _) = true
   | f(n, m: int) = if n mod m = 0 then false
                 else if m*m >= n then true
                 else f(n, m+1);
fun g(n: int): bool = f(n, 2);
fun h(n,m) = n;
```

Your tasks are the following:

1. Describe what function `g` does, in particular, discuss termination.
2. Evaluate the expressions *a)* `h(g(3),g(4))`, *b)* `h(g(1),g(0))`, and *c)* `h(g(2),g(0))` from the given definitions.

Solution:

1. Function `g` checks whether the integer argument is a prime number or not, in case $2 \leq n$. It returns `true` if it is, `false` if it is not. If $n = 1$ then we raise an exception and if $n = 0$ we have an infinite loop.
2. *a)* `h(g(3),g(4))` will evaluate to true, since we have no termination issues and 3 is a prime number.
   b) `h(g(1),g(0))` raises `RandomException`.
   c) `h(g(2),g(0))` goes into an infinite loop due to the call-by-value nature of SML and its attempt to evaluate `g` for 0.

Quiz 8: Codes(Given Nov. 15. 2010)

6pt

Problem 8.1 (Character code)

Consider the alphabets $A := \{x, y, z, t\}$ and $B := \{(,), :, |\}$ and the following function: $c\colon A \mapsto B^+$ with:

$$c(x) =:)$$
$$c(y) =:))$$
$$c(z) = | :$$
$$c(t) = (:))$$

1. Is c a character code? Explain.
2. Check whether c is a prefix code and, if it is not, modify it so that it is a prefix code.

Solution:

1. c is a character code by deinition.
2. c is not a prefix code ($c(x)$ is a prefix of $c(y)$. Therefore we modify it: $c(x) =:)($

Problem 8.2 (Formal language) 6pt

Give the definition of the formal language of the words over $\{0,1\}$ that are palindromes.

Note: A palindrome is a word w that is identical to w reversed. For example: "1001" and "0010100".

Solution: $L := \{0x0 \mid x \in L\} \cup \{1x1 \mid x \in L\}$

Quiz 9: Boolean Expressions(Given Nov. 22. 2010)

12pt

Problem 9.1 (Evaluating Expressions)

Given the expression E:= $\overline{(x_1 + x_2) * \overline{x_3}}$ Your tasks are:

1. What is the depth of the expression?
2. If $\varphi := ([\mathsf{T}/x_1], [\mathsf{F}/x_2], [\mathsf{T}/x_3])$ evaluate the expression using the evaluation function $\mathcal{I}_\varphi(E)$ and showing the whole computation.
3. Write down the truth table for the expression.

Solution:

1. The depth of the expression is 4.
2.

$$
\begin{aligned}
& \mathcal{I}_\varphi(\overline{(x_1 + x_2) * \overline{x_3}}) \\
= & \neg(\mathcal{I}_\varphi((x_1 + x_2) * \overline{x_3})) \\
= & \neg(\mathcal{I}_\varphi(x_1 + x_2) \wedge \mathcal{I}_\varphi(\overline{x_3})) \\
= & \neg((\mathcal{I}_\varphi(x_1) \vee \mathcal{I}_\varphi(x_2)) \wedge (\neg(\mathcal{I}_\varphi(x_3)))) \\
= & \neg((\varphi(x_1) \vee \varphi(x_2)) \wedge (\neg(\varphi(x_3)))) \\
= & \neg((\mathsf{T} \vee \mathsf{F}) \wedge (\neg(\mathsf{T}))) \\
= & \neg(\mathsf{T} \wedge \mathsf{F}) \\
= & \neg(\mathsf{F}) \\
= & \mathsf{T}
\end{aligned}
$$

3. The truth table is:

assignments			intermediate results			full
x_1	x_2	x_3	$x_1 + x_2$	$\overline{x_3}$	$(x_1 + x_2) * \overline{x_3}$	E
F	F	F	F	T	F	T
F	F	T	F	F	F	T
F	T	F	T	T	T	F
F	T	T	T	F	F	T
T	F	F	T	T	T	F
T	F	T	T	F	F	T
T	T	F	T	T	T	F
T	T	T	T	F	F	T

Quiz 10: Computing CNF and DNF(Given Nov. 29. 2010)

12pt

Problem 10.1 (CNF and DNF)
Find the CNF and DNF of the boolean function that corresponds to the expression:

$$x_2 + \overline{(x_1 + x_4) * (x_2 + x_3)} * x_4$$

Solution:
Item We infer CNF and DNF from the expression's truth table:

x_1	x_2	x_3	x_4	f	monomials	clauses
1	1	1	1	1	$x_1^1\, x_2^1\, x_3^1\, x_4^1$	
1	1	1	0	1	$x_1^1\, x_2^1\, x_3^1\, x_4^0$	
1	1	0	1	1	$x_1^1\, x_2^1\, x_3^0\, x_4^1$	
1	1	0	0	1	$x_1^1\, x_2^1\, x_3^0\, x_4^0$	
1	0	1	1	0		$x_1^0 + x_2^1 + x_3^0 + x_4^0$
1	0	1	0	0		$x_1^0 + x_2^1 + x_3^0 + x_4^1$
1	0	0	1	1	$x_1^1\, x_2^0\, x_3^0\, x_4^1$	
1	0	0	0	0		$x_1^0 + x_2^1 + x_3^1 + x_4^1$
0	1	1	1	1	$x_1^0\, x_2^1\, x_3^1\, x_4^1$	
0	1	1	0	1	$x_1^0\, x_2^1\, x_3^1\, x_4^0$	
0	1	0	1	1	$x_1^0\, x_2^1\, x_3^0\, x_4^1$	
0	1	0	0	1	$x_1^0\, x_2^1\, x_3^0\, x_4^0$	
0	0	1	1	0		$x_1^1 + x_2^1 + x_3^0 + x_4^0$
0	0	1	0	0		$x_1^1 + x_2^1 + x_3^0 + x_4^1$
0	0	0	1	1	$x_1^0\, x_2^0\, x_3^0\, x_4^1$	
0	0	0	0	0		$x_1^1 + x_2^1 + x_3^1 + x_4^1$

DNF: $x_1^1\, x_2^1\, x_3^1\, x_4^1 + x_1^1\, x_2^1\, x_3^1\, x_4^0 + x_1^1\, x_2^1\, x_3^0\, x_4^1 + x_1^1\, x_2^1\, x_3^0\, x_4^0 + x_1^1\, x_2^0\, x_3^0\, x_4^1 + x_1^0\, x_2^1\, x_3^1\, x_4^1 +$
$x_1^0\, x_2^1\, x_3^1\, x_4^0 + x_1^0\, x_2^1\, x_3^0\, x_4^1 + x_1^0\, x_2^1\, x_3^0\, x_4^0 + x_1^0\, x_2^0\, x_3^0\, x_4^1$

CNF: $(x_1^0 + x_2^1 + x_3^0 + x_4^0)\,(x_1^0 + x_2^1 + x_3^0 + x_4^1)\,(x_1^0 + x_2^1 + x_3^1 + x_4^1)\,(x_1^1 + x_2^1 + x_3^0 + x_4^0)$
$(x_1^1 + x_2^1 + x_3^0 + x_4^1)\,(x_1^1 + x_2^1 + x_3^1 + x_4^1)$
