Assignments for General CS 2 (320201)

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For Course Purposes Only

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Contents

Quiz 1: Representations(Given Sep. 8.)

12pt

Problem 1.1 (Representations)
An essential concept in computer science is the Representation.

- What is the intuition behind the term "representation"?
- Why do we need representations?
- Give an everyday example of a representation.

Solution:

- A representation is the realization of real or abstract persons, objects, circumstances, Events, or emotions in concrete symbols or models. This can be by diverse methods, e.g. visual, aural, or written; as three-dimensional model, or even by dance.
- we should always be aware, whether we are talking about the real thing or a representation of it. Allows us to abstract away from unnecessary details. Easy for computer to operate with
- e.g. graph is a representation of a maze from the lecture notes

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Quiz 2: Unary natural numbers(Given Sep. 15.)

6pt

Problem 2.1 (Peano's induction axiom)
State Peano's induction axiom and discuss what it can be used for.

Solution: Peano's induction axiom: Every unary natural number possesses property P , if

- the zero has property P and
- the successor of every unary natural number that has property P also possesses property P

Peano's induction axiom is useful to prove that all natural numbers possess some property. In practice we often use the axiom to prove useful equalities that hold for all natural numbers (e.g. binomial theorem, geometric progression).

6pt

Problem 2.2 (Natural numbers)
Prove or refute that $s(s(o))$ and $s(s(s(o)))$ are unary natural numbers and that their successors are different.

Solution:
Proof: We will prove the statement using the Peano axioms:

P.1 o is a unary natural number (axiom P1)

P.2 $s(o)$ is a unary natural number (axiom P2 and 1.)

P.3 $s(s(o))$ is a unary natural number (axiom P2 and 2.)

P.4 $s(s(s(o)))$ is a unary natural number (axiom P2 and 3.)

P.5 Since $s(s(s(o)))$ is the successor of $s(s(o))$ they are different unary natural numbers (axiom P2)

P.6 Since $s(s(s(o)))$ and $s(s(o))$ are different unary natural numbers their successors are also different (axiom P4 and 5.)

□

Quiz 3: Relation properties(Given Sep. 22.)

12pt

Problem 3.1 (Relation Properties)

a) Given the two sets $A = \{a, b, c, d\}$ and $B = \{1, 2, 3\}$, give an example set for each of the following relations:

1. $R_1 \subseteq A \times A$:
   $R_1 \neq \emptyset \wedge (\forall x, y \in A.(\langle x, y\rangle \in R_1 \wedge \langle y, x\rangle \in R_1) \Rightarrow x = y)$
2. $R_2 \subseteq A \times B$:
   $\forall x \in A.(\exists y \in B.\langle x, y\rangle \in R_2) \wedge (\forall z \in B.\langle x, z\rangle \in R_2 \Rightarrow (\nexists w \in B.w \neq z \wedge \langle x, w\rangle \in R_2))$
3. $R_3 \subseteq A \times B$:
   $\forall x \in A.\forall y \in B.(\exists z \in A.\langle x, z\rangle \in R_1 \wedge \langle z, y\rangle \in R_2) \Rightarrow \langle x, y\rangle \in R_3$

b) For each of the expressions above write what it represents.

Solution:

1. asks for a nonempty antisymmetric relation $R_1 \subseteq A \times A$. A possible solution is $R_1 = \{\langle a, b\rangle, \langle b, c\rangle, \langle c, d\rangle, \langle d, d\rangle\}$.
2. asks for a total function $f \colon A \to B$, so a possible solution is $R_2 = \{\langle a, 1\rangle, \langle b, 2\rangle, \langle c, 3\rangle, \langle d, 1\rangle\}$
3. asks for a composition of R_1 and R_2. In case of R_1, R_2 as defined above, $R_3 = R_2 \circ R_1 = \{\langle a, 2\rangle, \langle b, 3\rangle, \langle c, 1\rangle, \langle d, 1\rangle\}$

Quiz 4: SML(Given Sep. 29.)

12pt

Problem 4.1 (Lists)
Write an SML function that returns the sum of the last 2 elements of a list of integers. If the list has less than 2 elements simply return the sum of all elements.

Solution:

```
fun sum_last_two nil = 0
  | sum_last_two (a::nil) = a
  | sum_last_two (a::b::nil) = a+b
  | sum_last_two (a::l) = sum_last_two l;

(* Test cases *)
val test1 = (sum_last_two [1,2,3,4,5,6]) = 11;
val test2 = (sum_last_two []) = 0;
val test3 = (sum_last_two [6]) = 6;
```

Quiz 5: Abstract data types(Given Oct. 6.)

12pt

Problem 5.1 (ADTs and Ground Constructor Terms)
Given the ADT

$$\langle \{\mathbb{A}, \mathbb{B}\}, M \rangle$$

write down the constructor declarations, i.e. the set M such that the following are ground constructor terms of the respective sorts:

ground constructor term	sort
$g(a, c(a, b))$	$\mathbb{B}$
$f(g(d(a, e)))$	$\mathbb{A}$
$c(b, f(g(b, a)))$	$\mathbb{A}$

Solution:
$M = \{[a\colon \mathbb{A}], [b\colon \mathbb{A}], [c\colon \mathbb{A} \times \mathbb{A} \to \mathbb{A}], [d\colon \mathbb{A} \to \mathbb{A}], [e\colon \mathbb{A}], [f\colon \mathbb{B} \to \mathbb{A}], [g\colon \mathbb{A} \times \mathbb{A} \to \mathbb{B}]\}$

Quiz 6: Abstract procedures(Given Oct. 13.)

Problem 6.1 (Datatype Conversion) 4pt

Convert the SML datatype of shapes to an abstract data type. Use unary natural numbers to represent integers.

```
datatype shape =
   Circle of int
 | Square of int
 | Triangle of int * int * int
```

Solution:

$$\langle\{\mathbb{N},\mathbb{S}\},\{[o\colon\mathbb{N}],[s\colon\mathbb{N}\to\mathbb{N}][\mathit{circle}\colon\mathbb{N}\to\mathbb{S}],[\mathit{square}\colon\mathbb{N}\to\mathbb{S}],[\mathit{triangle}\colon\mathbb{N}\times\mathbb{N}\times\mathbb{N}\to\mathbb{S}]\}\rangle$$

Problem 6.2 ((Non)Terminating Abstract Procedures) 8pt

Write an abstract procedure τ on natural numbers that terminates on all odd arguments and does not terminate on even arguments. The ADT for unary natural numbers is provided below for your convenience.

$$\langle\{\mathbb{N}\},\{[o\colon\mathbb{N}],[s\colon\mathbb{N}\to\mathbb{N}]\}\rangle$$

Solution:

$$\langle\tau{::}\mathbb{N}\to\mathbb{N}\,;\,\{\tau(o)\rightsquigarrow\tau(s(s(o))),\tau(s(o))\rightsquigarrow o,\tau(s(s(n)))\rightsquigarrow\tau(n)\}\rangle$$

Quiz 7: Call by value(Given Nov. 20.)

12pt

Problem 7.1 (Call by Value)

Explain the concept of a "call-by-value" programming language in terms of evaluation order. Give an example program where this affects evaluation and termination, explain it.

Solution: A "call-by-value" programming language is one, where the arguments are all evaluated before the defining equations for the function are applied. As a consequence, an argument that contains a non-terminating call will be evaluated, even if the function ultimately disregards it. For instance, evaluation of the last line does not terminate.

```
fun myif (true,A,_) = A | myif (false,_,B) = B
fun bomb (n) = bomb(n+1)
myif(true,1,bomb(1))
```

Quiz 8: Codes and lexical ordering(Given Nov. 3.)

Problem 8.1 (Codes) 6pt

Let s be a string of length 3 that is a prefix of your first name.

1. Write down s.
2. Write down an alphabet A that contains all letters from s.
3. Give an example for a binary character code c with $\mathbf{dom}(c) = A$.
4. Encode s using the extension of c.

Solution: Name: Dimitar

- $s = Dim$
- $A := \{D, i, m\}$
- $c := \{\langle D, 0\rangle, \langle i, 1\rangle, \langle m, 00\rangle\}$
- $c'(Dim) = 0100$

Problem 8.2 (Lexical ordering) 6pt

Let $A := \{x, :, +, R\}$ and $\prec$ be the ordering relation on A. If $\prec_{\text{lex}}$ is the lexical ordering induced by $\prec$ and

$$RRRR :\prec_{\text{lex}} RR + RRx \prec_{\text{lex}} RR : RR \prec_{\text{lex}} xRRRxR$$

write down one possibility for $\prec$.

Solution:

$R \prec + \prec : \prec x$ OR
$R \prec + \prec x \prec :$ OR
$R \prec x \prec + \prec :$

Quiz 9: Boolean Expressions(Given Nov. 10.)

12pt

Problem 9.1 (Evaluating Expressions)
Given $\varphi_1 := ([\mathsf{T}/x_1], [\mathsf{F}/x_2], [\mathsf{F}/x_3])$ and $\varphi_2 := ([\mathsf{F}/x_1], [\mathsf{T}/x_2], [\mathsf{F}/x_3])$ write a boolean expression containing the variables x_1, x_2 and x_3 that evaluates to T under φ_1 and to F under φ_2. Show the evaluation of your expression under the two assignments step by step.

Solution: Expression: $x_1 * (\overline{x_2} * \overline{x_3})$

$$\begin{aligned}
\mathcal{I}_\varphi(x_1 * (\overline{x_2} * \overline{x_3})) &= \mathcal{I}_\varphi(x_1) \wedge \mathcal{I}_\varphi(\overline{x_2} * \overline{x_3}) \\
&= \varphi_1 x_1 \wedge \mathcal{I}_\varphi(\overline{x_2}) \wedge \mathcal{I}_\varphi(\overline{x_3}) \\
&= \mathsf{T} \wedge \neg(\mathcal{I}_\varphi(x_2)) \wedge \neg(\mathcal{I}_\varphi(x_3)) \\
&= \mathsf{T} \wedge \neg(\varphi_1 x_2) \wedge \neg(\varphi_1 x_3) \\
&= \mathsf{T} \wedge \neg(\mathsf{F}) \wedge \neg(\mathsf{F}) \\
&= \mathsf{T} \wedge \mathsf{T} \wedge \mathsf{T} = \mathsf{T} \\
\mathcal{I}_\varphi(x_1 * (\overline{x_2} * \overline{x_3})) &= \mathcal{I}_\varphi(x_1) \wedge \mathcal{I}_\varphi(\overline{x_2} * \overline{x_3}) \\
&= \varphi_2 x_1 \wedge \mathcal{I}_\varphi(\overline{x_2}) \wedge \mathcal{I}_\varphi(\overline{x_3}) \\
&= \mathsf{F} \wedge \neg(\mathcal{I}_\varphi(x_2)) \wedge \neg(\mathcal{I}_\varphi(x_3)) \\
&= \mathsf{F} \wedge \neg(\varphi_2 x_2) \wedge \neg(\varphi_2 x_3) \\
&= \mathsf{F} \wedge \neg(\mathsf{T}) \wedge \neg(\mathsf{F}) \\
&= \mathsf{F} \wedge \mathsf{F} \wedge \mathsf{T} = \mathsf{F}
\end{aligned}$$

Quiz 10: Normal forms and Landau sets(Given Nov. 17.)

6pt

Problem 10.1 (CNF and DNF)
Write the CNF and DNF of the boolean function that corresponds to the truth table below.

x_1	x_2	x_3	f
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	0

Solution:

DNF: $\overline{x_1}\,\overline{x_2}\,\overline{x_3} + \overline{x_1}\,x_2\,\overline{x_3} + x_1\,\overline{x_2}\,\overline{x_3} + x_1\,\overline{x_2}\,x_3 + x_1\,x_2\,\overline{x_3}$

CNF: $(x_1 + x_2 + \overline{x_3})\,(x_1 + \overline{x_2} + \overline{x_3})\,(\overline{x_1} + \overline{x_2} + \overline{x_3})$

6pt

Problem 10.2 (Landau sets)
Order the landau sets below by specifying which ones are subsets and which ones are equal (e.g.: $O(a) \subset O(b) \subset O(c) \equiv O(d) \subset O(e) \ldots$)

$$O(n^2);\ O((n)!);\ O(|\sin n|);\ O(n^n);\ O(1);\ O(2^n);\ O(2n^2 + 2^{72})$$

Solution: $O(|\sin n|) \subset O(1) \subset O(2n^2 + 2^{72}) \equiv O(n^2) \subset O(2^n) \subset O((n)!) \subset O(n^n)$

Quiz 11: KV maps(Given Nov. 24.)

12pt

Problem 11.1 (Karnaugh-Veitch Minimization)
Use a KV map to determine the minimal polynomial for the following truth table:

#	A	B	C	D	V
0	F	F	F	F	T
1	F	F	F	T	F
2	F	F	T	F	T
3	F	F	T	T	F
4	F	T	F	F	F
5	F	T	F	T	F
6	F	T	T	F	F
7	F	T	T	T	T
8	T	F	F	F	T
9	T	F	F	T	T
10	T	F	T	F	T
11	T	F	T	T	T
12	T	T	F	F	T
13	T	T	F	T	T
14	T	T	T	F	T
15	T	T	T	T	T

Solution: The KV map looks like this:

	$\overline{A}\overline{B}$	$\overline{A}B$	AB	$A\overline{B}$
$\overline{C}\overline{D}$	T	F	T	T
$\overline{C}D$	F	F	T	T
CD	F	T	T	T
$C\overline{D}$	T	F	T	T

The minimal polynomial is: $A + \overline{B}\,\overline{D} + B\,C\,D$

Quiz 12: Calculi and boolean expressions(Given Dec. 1.)

6pt

Problem 12.1 (Calculus Properties)
Explain briefly what the following properties of calculi mean:

- correctness
- completeness

Solution:

- correctness ($\mathcal{H} \vdash \mathbf{B}$ implies $\mathcal{H} \models \mathbf{B}$) - A calculus is correct if any derivable(provable) formula is also a valid formula.
- completeness ($\mathcal{H} \models \mathbf{B}$ implies $\mathcal{H} \vdash \mathbf{B}$) - A calculus is complete if any valid formula can also be derived(proven).

6pt

Problem 12.2 (Properties of boolean expressions)
In the table below mark the properties that each of the expressions has.

	valid	satisfiable	falsifiable	unsatisfiable
$(x_1 + x_2) * \overline{x_1}$				
$(x_1 + x_2) * (\overline{x_1} * \overline{x_2})$				
$x_1 * x_2 + (\overline{x_1} + \overline{x_2})$				

Solution:

	valid	satisfiable	falsifiable	unsatisfiable
$(x_1 + x_2) * \overline{x_1}$		yes	yes	
$(x_1 + x_2) * (\overline{x_1} * \overline{x_2})$			yes	yes
$x_1 * x_2 + (\overline{x_1} + \overline{x_2})$	yes	yes		