Assignments for General CS 1 (320201)

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For Course Purposes Only

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Contents

Quiz 1: Algorithms(Given Sep. 10.)

12pt

Problem 1.1 (Algorithms)
One of the most essential concepts in computer science is the Algorithm.

- What is the intuition behind the term “algorithm”.
- What determines the quality of an algorithm?
- Give an everyday example of an algorithm.

Solution:

- An algorithm is a series of instructions to control a (computation) process.
- Termination, correctness, performance
- e. g. a recipe

Quiz 2: Relations and Functions(Given Sep. 18.)

6pt

Problem 2.1 (Relation Properties)
Given a base set $A := \{a, b, c\}$, for each of the following properties:

1. reflexive
2. symmetric
3. antisymmetric
4. transitive

write down a relation $R_i \subseteq A \times A$ ($i = 1, \ldots, 4$) that has property i and contains at least three tuples.

Solution:

1. $R_1 := \{\langle a, a\rangle, \langle b, b\rangle, \langle c, c\rangle\}$ (no other possibility)
2. e. g. $R_2 := \{\langle a, b\rangle, \langle b, c\rangle, \langle b, a\rangle, \langle c, b\rangle\}$
3. e. g. $R_3 := \{\langle a, b\rangle, \langle b, c\rangle, \langle a, c\rangle\}$
4. e. g. $R_4 := \{\langle a, b\rangle, \langle b, c\rangle, \langle a, c\rangle\}$

6pt

Problem 2.2 (Function Definition)
Let A and B be sets. State the definition of the concept of a partial function with domain A and codomain B. Also state the definition of a total function with domain A and codomain B.

Solution: Let A and B be sets, then a relation $R \subseteq AB$ is called a **partial/total function**, iff for each $a \in A$, there is at most/exactly one $b \in B$, such that $\langle a, b\rangle \in R$.

Quiz 3: Properties of Functions(Given Sep. 24.)

12pt

Problem 3.1: For the functions below determine whether they are injective, surjective, bijective. If they are not, give counter-examples. Give the inverse function for any bijective one.

$f\colon \mathbb{N} \to \mathbb{N}$	$x \mapsto 5x$
$h\colon \mathbb{N} \to \mathbb{N}$	$x \mapsto x\ mod\ 7$
$i : \{2,3,4\} \to \{1,2,3\}$	$x \mapsto x - 1$

Solution:

- f is injective and not surjective (e.g. $4 \notin f(\mathbb{N})$), so f is not bijective
- h is not injective (e.g. $h(10) = h(3)$), not surjective (e.g. $10 \notin f(\mathbb{N})$), and thus not bijective
- i is injective, surjective and bijective; $i^{-1}(x) := x + 1$

Quiz 4: SML(Given Oct. 8.)

6pt

Problem 4.1: Write down the type (with explicit brackets) of the following expressions `fn (x:int) => (fn (y) => x:`

6pt

Problem 4.2: Define the `member` relation which checks whether an integer is member of a list of integers. The solution should be a function of type `int * int list -> bool`, which evaluates to `true` on arguments `n` and `l`, iff `n` is an element of the list `l`.

Solution: The simplest solution is the following

```
fun member(n,nil) = false
  | member(n,h::r) = if n=h then true else member(n,r);
```

The intuition here is that a is a member of a list l, iff it is the first element, or it is a member of the rest list.

Note that we cannot just use `member(n,n::r)` to eliminate the conditional, since SML does not allow duplicate variables in matching. But we can simplify the conditional after all: we can make use of SML's `orelse` function which acts as a logical "or" and get the slightly more elegant program

```
fun member(n,nil) = false
  | member(n,h::r) = (n=h) orelse member(n,r);
```

Quiz 5: Abstract data types(Given Oct. 15.)

6pt

Problem 5.1 (SML datatypes vs Abstract Data Types)
Given the SML datatypes

1. `datatype A = a | f of A * A`
2. `datatype B = b | g of A -> B`

Write down one abstract data type in math notation representing both SML datatypes at once.

Solution:

$$\langle\{\mathbb{A},\mathbb{B}\},\{[a\colon\mathbb{A}],[b\colon\mathbb{B}],[f\colon\mathbb{A}\times\mathbb{A}\to\mathbb{A}],[g\colon\mathbb{A}\to\mathbb{B}\to\mathbb{B}]\}\rangle$$

6pt

Problem 5.2 (Substitution)
Apply the substitutions $\sigma := [b/x],[(g(a))/y],[a/w]$ and $\tau := [(h(c))/x],[c/z]$ to the terms $s := f(g(x,g(a,x,b),y))$ and $t := g(x,x,h(y))$ (give the 4 result terms $\sigma(s)$, $\sigma(t)$, $\tau(s)$, and $\tau(t)$).

Solution:

$$\begin{array}{rclcrcl} \sigma(s) & = & f(g(a,f(b),g(a,a,b))) & & \sigma(t) & = & g(a,f(b),h(a)) \\ \tau(s) & = & f(g(f(b),y,g(a,f(b),b))) & & \tau(t) & = & g(f(b),y,h(c)) \end{array}$$

Quiz 6: Abstract procedures(Given Oct. 22.)

12pt

Problem 6.1 (Recursion and termination)
Given the following SML function,

```
fun f(0) = true
  | f(1) = false
  | f(2) = false
  | f(n) = f(n-3)
```

1. describe in your own words what this function computes.
2. give the recursion relation of the corresponding abstract procedure $\mathcal{P}$.
3. Does $\mathcal{P}$ terminate on all natural number inputs? (Please justify your claim!)

Solution: Thanks to Anca Dragan for suggesting this exercise.

1. The function computes whether a given natural number is divisible by 3.
2. $\langle n, n-3 \rangle$
3. Yes, would prove this by induction with the base cases 0, 1, and 2, and the induction step $n-3 \mapsto n$.

Quiz 7: Codes(Given Nov. 5.)

12pt

Problem 7.1: Let $A := \{\mathtt{a}, \mathtt{h}, /, \#, \mathtt{x}\}$ and $\prec$ be the ordering relation on A with $\mathtt{x} \prec \# \prec / \prec \mathtt{h} \prec \mathtt{a}$. Order the following strings in A^* in the lexical order $<_{\text{lex}}$ induced by $\prec$.

$s_1 = \#\#\#\#$	$s_2 = \#\#\mathtt{x}\#\#\mathtt{h}$	$s_3 = \epsilon$
$s_4 = \#\#\mathtt{h}\#\#\mathtt{x}$	$s_5 = \mathtt{a}\#\#\#\mathtt{a}\#$	$s_6 = \#\#\#\#/$

Quiz 8: Boolean Algebra(Given Nov. 12.)

12pt

Problem 8.1: Determine the Boolean function f_e induced by the Boolean expression $e := ((x1 + x2) * \overline{x2 * x3})$. Moreover determine the CNF and DNF of f_e.

Quiz 9: Landau sets(Given Nov. 19.)

12pt

Problem 9.1: Determine for the following functions f and g whether $f \in O(g)$, or $f \in \Omega(g)$, or $f \in \Theta(g)$, explain your answers.

f	g	f	g
4572	84	$n^3 + 3 * n$	n^3
$\log(n^3)$	$\log(n)$	$n^2 - 2^2$	n^3
16^n	2^n	n^n	2^{n+1}

Solution: The following table summarizes the results.

Fact	Explanation
$4572 \in \Theta(84)$	For all $n \in \mathbb{N}$ we have $1000 \cdot 84 \leq 4572$ and $0.001 \cdot 4572 \leq 84$
$n^3 + 3 * n \in \Omega(n^3)$	For all n: If $c = 1$ then $n^3 + 3 * n \geq n^3$ and if $c = 10$ then $n^3 + 3 * n \leq n^3$.
$(logn^3) \in \Theta(log(n))$	Since $log(n^3) = 3 \cdot log(n)$
$n^2 - 2^2 \in \Omega(n^3)$	larger exponents win
$16^n \in \Theta(2^n)$	For all c there is an n such that $16^n \geq c \cdot 2^n$; just take n for a given c such that $8^n \geq c$.
$n^n \in O(2^{n+1})$	For $c = 2$ and $n > 1$ we have $2^{n+1} = 2 * 2^n \leq c \cdot n^n$

Quiz 10: Karnaugh-Veitch Diagrams(Given Nov. 26.)

12pt

Problem 10.1 (Karnaugh-Veitch Minimization)
Use a KV map to determine the minimal polynomial of the following function:

$x1$	$x2$	$x3$	$x4$	f
F	F	F	F	T
F	F	F	T	F
F	F	T	F	T
F	F	T	T	F
F	T	F	F	T
F	T	F	T	T
F	T	T	F	F
F	T	T	T	F
T	F	F	F	T
T	F	F	T	F
T	F	T	F	T
T	F	T	T	F
T	T	F	F	T
T	T	F	T	T
T	T	T	F	F
T	T	T	T	T

Solution: Note that the values in the four corners form one group! To obtain a *minimal* polynomial, one must also use intersecting groups here.

Quiz 11: Calculi(Given Dec. 3.)

12pt

Problem 11.1: We have proven the correctness of the Hilbert calculus $\mathcal{H}^0$ in class. The problems of this quiz is about two incorrect calculi $\mathcal{C}^1$ and $\mathcal{C}^2$ which differ only slightly from $\mathcal{H}^0$.

What makes them incorrect?

Hint: The fact that $\mathcal{H}^0$ has two axioms, but each of $\mathcal{C}^1$ and $\mathcal{C}^2$ only have one is not the point. Remember the properties of axioms and inference rules which are preconditions for a correct calculus.

Why is this calculus $\mathcal{C}^1$ incorrect?

- $\mathcal{C}^1$ Axiom:$P \Rightarrow P \wedge Q$
- $\mathcal{C}^1$ Inference Rules: $\dfrac{\mathbf{A} \Rightarrow \mathbf{B} \quad \mathbf{A}}{\mathbf{B}}\,\mathrm{MP}$ $\qquad \dfrac{\mathbf{A}}{[\mathbf{B}/P]\mathbf{A}}\,\mathrm{Subst}$

Why is this calculus $\mathcal{C}^2$ incorrect?

- $\mathcal{C}^2$ Axiom: $P \Rightarrow Q \Rightarrow P$
- $\mathcal{C}^2$ Inference Rules: $\dfrac{\mathbf{A} \vee \mathbf{B} \quad \mathbf{A}}{\mathbf{A} \wedge \mathbf{B}}\,R2$ $\qquad \dfrac{\mathbf{A}}{[\mathbf{B}/P]\mathbf{A}}\,\mathrm{Subst}$

Solution: A correct calculus requires valid axioms.
However the Axiom of $\mathcal{C}^1$ is not valid since the assignment $\varphi = [T/P], [T/Q], [F/R]$ makes it false.
