Name: Matriculation Number:

Midterm Exam
General CS (CH08-320101)

November 10, 2016

You have 75 minutes(sharp) for the test;
Write the solutions to the sheet.

The estimated time for solving this exam is 68 minutes, leaving you 7 minutes for revising your exam.

You can reach 55 points if you solve all problems. You will only need 50 points for a perfect score, i.e. 5 points are bonus points.

Different problems test different skills and knowledge, so do not get stuck on one problem.

	To be used for grading, do not write here										
prob.	1.1	1.2	2.1	2.2	2.3	3.1	4.1	5.1	6.1	Sum	grade
total	2	7	6	2	4	12	9	8	5	55	
reached											

Please consider the following rules; otherwise you may lose points:

- Always justify your statements. Unless you are explicitly allowed to, do not just answer “yes” or “no”, but instead prove your statement or refer to an appropriate definition or theorem from the lecture.
- If you write program code, give comments, so that we can award you partial credits!

1 GenCS Classics and Induction

Problem 1.1 (Greek Alpabet)

Fill in the blanks in the table of Greek letters. Note that capitalized names denote capital Greek letters. 2pt 2min

Symbol	θ	τ	ν	ι				
Name					gamma	chi	xi	rho

Solution:

Symbol	θ	τ	ν	ι	γ	χ	ξ	ρ
Name	theta	tau	nu	iota	gamma	chi	xi	rho

Problem 1.2 (De Moivre's formula)

Prove by induction or refute that the following equation holds: 7pt 10min

$$(i\ \sin(x) + \cos(x))^n = i\ \sin(nx) + \cos(nx)$$

where $x \in \mathbb{C}$, $n \in \mathbb{N}$ and i is the imaginary unit, i.e. $i^2 = -1$.

Hint: You can use the following trigonometric identities:

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$
$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

Solution:

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2 Relations and Functions

Problem 2.1 (Properties of Relations)

Suppose that R and S are equivalence relations on a set A. Prove or refute by a counterexample each of these statements: 6pt 7min

1. $R \cap S$ is an equivalence relation
2. $R \cup S$ is an equivalence relation
3. $S \circ R$ is irreflexive
4. $R \backslash S$ is an equivalence relation

Solution:

1. $R \cap S$ is an equivalence relation as
   - $(a, a)\ \forall a \in A$ (reflexivity)
   - if $(a, b) \in (R \cap S)$ then $(b, a) \in (R \cap S)$ as both R and S are symmetric (symmetry)
   - if $\{(a, b), (b, c)\} \in (R \cap S)$ then $(a, c) \in (R \cap S)$ as both R and S are transitive (transitivity)
2. $R \cup S$ is not an equivalence relation.
   Counterexample: let us take $A = \{a, b, c\}$, $R = \{(a, b), (b, a), (a, a), (b, b), (c, c)\}$ and $S = \{(b, c), (c, b), (a, a), (b, b), (c, c)\}$.
   $\{(a, b), (b, c)\} \in (R \cup S)$ but $(a, c) \notin (R \cup S)$.

3. $S \circ R$ is not irreflexive.
   Counterexample: let us consider $A = \{a\}$, $R = \{(a,a)\}$ and $S = \{(a,a)\}$.
   $(a,a) \in (S \circ R)$
4. $R \backslash S$ is not an equivalence relation as $\forall a \in A\ (a,a) \notin (R \backslash S)$

Problem 2.2 (Function Definition)

Let A and B be sets. State the definition of the concept of a partial function with domain A and codomain B. Also state the definition of a total function with domain A and codomain B. 2pt 2min

Solution: Let A and B be sets, then a relation $R \subseteq AB$ is called a **partial/total function, iff for each $a \in A$, there is at most/exactly one $b \in B$, such that $(a,b) \in R$.**

Problem 2.3 (Function injectivity/surjectivity)

4pt 7min

1. Prove or refute that the function $f: \mathbb{N} \to \mathbb{N}; n \mapsto 2n+1$ is bijective.
2. Prove or refute that the function $g: \mathbb{N} \to \mathbb{N}\backslash\{0\}; n \mapsto n+1$ is bijective.

Solution:
1. f is not bijective it is not surjective, For example $2 \in \mathbb{N}$ and there is no $n \in \mathbb{N}$ that $2 = 2n+1$.
2. g is bijective: Let $n \in (\mathbb{N}\backslash\{0\})$, then $n = g(n-1)$, so g is surjective. It is also injective by the third Peano axiom.

3 Abstract Data Types and Abstract Procedures

Problem 3.1 (ADT Tower game)

12pt 18min

Jacob is thinking to create a new video game in order to impress his GenCS classmates. To plan it thoroughly, he decided to firstly represent his game using an abstract data type. He is thinking about a tower defense game, in which an army tries to attack a city which is protected by a single tower. The **tower** needs to store the value of its hit points, represented by a natural number. The army is composed of only **knights** and **archers** and each of them will attack the tower only once in a battle.
To improve their combat abilities, before the battle each soldier visits the local wizard, which uses his magic to strengthen their attack with a special power. The wizard's magic can affect each soldier differently: some of them are not affected in any way, some get a special power based on **fire** and others get a special power based on **ice**.

Note: Assume that an ADT for natural numbers is given (so use them normally, in decimal notation) and it comes with a pre-defined procedure for addition.

1. Help Jacob by writing an ADT which can represent any battle in his game: In a battle there must exist a single tower and an attacking army composed of any number (≥ 0) of soldiers.
2. Give the representation of a battle in which a tower with 200 hit points is attacked by an army of 3 soldiers: an archer with no special power, a knight with power based on fire and an archer with power based on ice.

3. The attack of each soldier can be measured by a damage value which is represented by a natural number and depends on the type of the soldier (base damage) and on the soldier's special power (bonus damage). The knights have a base damage of 50 and the archers have a base damage of 20. The fire bonus doubles the base damage of a soldier and the ice bonus adds 30 to it. Create an abstract procedure named towerDestroyed which, given a battle, determines whether the tower will be destroyed after attack of all soldiers. Assume that the ADT for booleans B is given.

Note: For the battle representation from above, the result of the procedure would be false.

Solution: $\langle\{Battle, Soldier, Power, \mathbb{N}\}, \{[knight\colon Power \to Soldier], [archer\colon Power \to Soldier], [none\colon Powe$

$$addS(archer(ice), addS(knight(fire), addS(archer(none), Tower(200))))$$

$$\langle less{::}\mathbb{N} \times \mathbb{N} \to \mathbb{B}\,;\ \left\{\begin{array}{l} less(0, s(x)) \rightsquigarrow True \\ less(s(a), s(b)) \rightsquigarrow less(a,b) \\ less(s(x), 0) \rightsquigarrow False \end{array}\right\}\rangle$$

$$\langle bonus{::}\mathbb{N} \times Power \to \mathbb{N}\,;\ \left\{\begin{array}{l} bonus(n, none) \rightsquigarrow n \\ bonus(n, fire) \rightsquigarrow +(n,n) \\ bonus(n, ice) \rightsquigarrow +(n, 30) \end{array}\right\}\rangle$$

$$\langle dmg{::}Soldier \to \mathbb{N}\,;\ \left\{\begin{array}{l} dmg(knight(n)) \rightsquigarrow bonus(50, n) \\ dmg(archer(n)) \rightsquigarrow bonus(20, n) \end{array}\right\}\rangle$$

$$\langle TowerHP{::}Battle \to \mathbb{N}\,;\ \left\{\begin{array}{l} TowerHP(Tower(n)) \rightsquigarrow n \\ TowerHP(addS(s,b)) \rightsquigarrow TowerHP(b) \end{array}\right\}\rangle$$

$$\langle SoldiersDmg{::}Battle \to \mathbb{N}\,;\ \left\{\begin{array}{l} SoldiersDmg(Tower(n)) \rightsquigarrow 0 \\ SoldiersDmg(addS(s,b)) \rightsquigarrow +(dmg(s), SoldiersDmg(b)) \end{array}\right\}\rangle$$

$$\langle towerDestroyed{::}Battle \to \mathbb{B}\,;\ \{\ towerDestroyed(battle) \rightsquigarrow less(TowerHP(battle), SoldiersDmg(battle)\ \}\rangle$$

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4 Programming in Standard ML

Problem 4.1 (SML Binary addition)

9pt
10min

On a unsigned byte one can retain integers from 0 to 255 by assigning 0 or 1 to each of the 8 bits. Binary addition has the following rules:

1. The procedure starts from right to left.
2. $0 + 1 = 1$
3. $1 + 0 = 1$
4. $0 + 0 = 0$
5. $1 + 1 = 0$ and you pass 1 to the next addition on the left.
6. One cannot represent numbers larger than 255.

Example:

$$00001101 + 00001100 = 00011001$$

Given two binary numbers given as strings, write a SML function that computes the binary addition and returns a string as an output. You can assume that the input will be valid.

Solution:

5 Formal Languages and Codes

Problem 5.1 (Code definitions)

8pt

7min

Define the following concepts and give an example of each:

1. Character code.
2. String code.
3. Prefix code.

Why are prefix codes also string codes? Give an intuitive explanation

Solution:

1. Let A and B be alphabets, then we call an injective function c from A to B^+ a character code
2. Let c' be a function from A^* to B^* if it is injective then it induces a string code
3. A (character) code $c : A \to B+$ is a prefix code if none of the codewords is a proper prefix to an other codeword

Proof of a prefix code being a string code in slide 130.

6 Boolean Algebra

Problem 6.1 (CNF and DNF)

Write the CNF and DNF of the boolean function that corresponds to the truth table below. 5pt

5min

x_1	x_2	x_3	f
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	0

Solution:

DNF: $\overline{x_1}\,\overline{x_2}\,\overline{x_3} + \overline{x_1}\,x_2\,\overline{x_3} + x_1\,\overline{x_2}\,\overline{x_3} + x_1\,\overline{x_2}\,x_3 + x_1\,x_2\,\overline{x_3}$

CNF: $(x_1 + x_2 + \overline{x_3})\,(x_1 + \overline{x_2} + \overline{x_3})\,(\overline{x_1} + \overline{x_2} + \overline{x_3})$