Name: Matriculation Number:

Midterm Exam 2
General CS I (320101)

November 12, 2013

You have 75 minutes(sharp) for the test;
Write the solutions to the sheet.

The estimated time for solving this exam is 0 minutes, leaving you 75 minutes for revising your exam.

You can reach 0 points if you solve all problems. You will only need 70 points for a perfect score, i.e. -70 points are bonus points.

Different problems test different skills and knowledge, so do not get stuck on one problem.

	To be used for grading, do not write here	
prob.	Sum	grade
total	0	
reached		

Please consider the following rules; otherwise you may lose points:

- Always justify your statements. Unless you are explicitly allowed to, do not just answer "yes" or "no", but instead prove your statement or refer to an appropriate definition or theorem from the lecture.
- If you write program code, give comments, so that we can award you partial credits!

1 GenCS Classics and Induction

Problem 1.1 (Greek Letters)

Fill in the blanks in the following table of Greek letters. Note that capitalized names denote capital Greek letters. 5pt

2min

Symbol		ψ	ν			Λ		ω	Ψ	
Name	Delta			sigma	delta		Xi			chi

Solution:

Symbol	Δ	ψ	ν	σ	δ	Λ	Ξ	ω	Ψ	χ
Name	Delta	psi	nu	sigma	delta	Lambda	Xi	omega	Psi	chi

Problem 1.2 (Giving a Prize in an Odd-Numbered Group)

The professor decides he is going to give a prize to a student. He heard about the following situation that works for an odd number of students: 10pt

8min

Take the students to the campus green, each of them with a ball. Let them walk around for 5 minutes. When this time is over each student will throw the ball to the student that is nearest to him. One student does not get a ball, this student receives the prize.

Prove by induction that this procedure always determines a unique winner if the number of student is odd and the distances between the students are all different.

Solution: Base Case: $n = 3$

Lets have people a, b, c. Since all the distances are different then we can order them without lost of generality as: $d_{ab} < d_{bc} < d_{ac}$ then since $d_{ab} < d_{bc}$, b will hit a and since $d_{ab} < d_{ac}$ then a will hit b and c will not be hit.

Step case: $n = k \to n = k + 2$

Since we have a finite number of positive numbers we can find the minimum of the set of distances. By the same logic used in the base case suppose $d_{p_1 p_2}$ is the minimum. Then p_1 is closest than anyone else to p_2 so he gets hit by him. Similarly for p_2. These two people are hit by each other so they do not influence the rest of the hitting distribution so we can ignore them. Now we are left with k people. Since we supposed the property held for k people then we know that there will be a person that does not get hit.

2 Relations and Functions

Problem 2.1 (Relation Properties)

10pt

8min

1. You are given the set $A := \{1, 2, 3, 4\}$ and the relation

$$R \subseteq A \times A, \quad R := \{\langle 1,1\rangle, \langle 2,2\rangle, \langle 3,3\rangle, \langle 4,4\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 2,3\rangle, \langle 3,2\rangle\}$$

Determine whether R is *a)* reflexive, *b)* symmetric, *c)* transitive, or *d)* antisymmetric. If the relation does not have a certain property, give a counter-example to show that.

2. How many relations are there over a set B with $\#(B) = n$ that are symmetric but *not* reflexive?

Solution:

1. *a)* The relation is obviously reflexive according to the definitions.
   b) and transitive as well
   c) It is not symmetric, because it doesn't contain $\langle 2,1\rangle$.
   d) It is not antisymmetric because, e.g. it has $\langle 2,3\rangle$ and $\langle 3,2\rangle$.

2. For the relation not to be reflexive, it has to not contain all the pairs $\langle x,x\rangle.x \in B$, but it can contain some of them. There are 2^n possibilities to choose from those pairs, from those in $2^n - 1$ cases it won't be reflexive. Since $\#(B \times B) = n^2$, there are $n^2 - n$ other pairs that we can choose to be in the relation. If we choose a pair, we have to also include its symmetric part for the relation to be symmetric. So the possibilities here are $\frac{n^2-n}{2}$. We now combine the 2 properties and we get that the asnwer is $(2^n - 1) \times 2^{\frac{n^2-n}{2}}$.

Problem 2.2 (Function properties)

10pt

10min

1. State in mathtalk the definition of an injective total function (you may not take the concept of a function as given; so your definition will have two steps).

2. Determine whether each of the following functions is injective, surjective or bijective:

   (a) $f\colon \mathbb{N} \to \mathbb{N}$ with $f(n) := 2n$

   (b) $g\colon \mathbb{N} \to \mathbb{N}$ with $g(n) := \lfloor \frac{n}{2} \rfloor$ (integer division)

   (c) $h\colon \mathbb{R} \to \mathbb{R}$ with $h(x) := \begin{cases} x^3 & \text{if } (x \geq 1) \\ (2)x & \text{if } (x < 1) \end{cases}$

Solution:

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3 Abstract Data Types and Abstract Procedures

Problem 3.1 (Solarsystems)

6pt

6min

1. Construct an ADT with a sort $\mathbb{S}$ for solar systems. A solar system consists of a sun, an arbitrary number of planets (either gas giants or terestrial planets) that each can have an arbitrary number of moons.

Hint: Feel free to use as many sorts as you need.

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2. Now represent our solar system using this ADT:

   Our solar system consists of the sun, the terestrial planets Mercury (no moon), Venus (no moon), Earth (1 moon) and Mars (2 moons) and the gas giants Jupiter (67 moons!), Saturn (47 moons), Uranus (27 moons) and Neptune (13 moons). Sadly, Pluto is not classified as a planet anymore, so in our solar system ADT it does not exist.

Solution: First the ADT:

$$\langle\{\mathbb{M},\mathbb{P},\mathbb{S}\},\{[m\colon\mathbb{M}],[g\colon\mathbb{P}],[t\colon\mathbb{P}],[sun\colon\mathbb{S}],[addMoon\colon\mathbb{M}\times\mathbb{P}\to\mathbb{P}],[addPlanet\colon\mathbb{P}\times\mathbb{S}\to\mathbb{S}]\}\rangle$$

And now the term:

$$\begin{aligned}&addPlanet(addMoon^{13}(m,g),addPlanet(addMoon^{27}(m,g),addPlanet(addMoon^{47}(m,g),\\&\quad addPlanet(addMoon^{63}(m,g),addPlanet(addMoon(m,addMoon(m,t)),\\&\quad addPlanet(addMoon(m,t),addPlanet(t,\\&\quad addPlanet(t,sun))))))))\end{aligned}$$

Problem 3.2 (ADTs and Substitutions)

Consider the following ADT $\mathcal{A}$ 10pt

6min

$$\langle\{\mathbb{A},\mathbb{B},\mathbb{C},\mathbb{D}\},\{[a\colon\mathbb{A}],[b\colon\mathbb{B}],[c\colon\mathbb{C}],[d\colon\mathbb{D}],[f\colon\mathbb{A}\times\mathbb{B}\times\mathbb{C}\to\mathbb{C}],[g\colon\mathbb{D}\to\mathbb{A}],[h\colon\mathbb{C}\times\mathbb{B}\to\mathbb{B}],[i\colon\mathbb{B}\to\mathbb{D}]\}\rangle$$

1. Which of the following expressions are terms over $\mathcal{A}$? what sorts are they (if they are terms)?

#	expression	term(Y/N)	sort
1	$g(i(b))$		
2	$f(g(d),h(c,h(d,b)),c)$		
3	$h(f(a,b,i(b)),b)$		
4	$i(h(f(a,b,c),h(c,b)))$		

   If the expressions are not terms, give a reason.

2. Now apply the substitutions $\sigma := ([d/x_{\mathbb{B}}],[f(a,h(c,b),c)/y_{\mathbb{C}}])$ and $\rho := ([i(h(c,b))/x_{\mathbb{D}}],[a/z_{\mathbb{A}}],[g(i(h(c,b)))/w_{\mathbb{A}}])$ to the following terms.

#	expression	σ instance	ρ instance
1	$g(i(x_{\mathbb{B}}))$		
2	$f(g(x_{\mathbb{D}}),f(a,b,y_{\mathbb{C}}),g(x_{\mathbb{D}}))$		
3	$i(h(x_{\mathbb{C}},y_{\mathbb{B}}))$		

Solution:

1.

#	expression	term(Y/N)	sort
1	$g(i(b))$	Y	$\mathbb{A}$
2	$f(g(d), h(c, h(d, b)), c)$	N	-
3	$h(f(a, b, i(b)), b)$	N	-
4	$i(h(f(a, b, c), h(c, b)))$	Y	$\mathbb{D}$

2.

#	expression	σ instance	ρ instance
1	$g(i(x_{\mathbb{B}}))$	$g(i(d))$	$g(i(h(c, b)))$
2	$f(g(x_{\mathbb{D}}), f(a, b, y_{\mathbb{C}}), g(x_{\mathbb{D}}))$	$f(g(d), i(f(a, h(c, b), c), g(w))$	$f(g(i(h(c, b))), i(y), g(g(i(h(c, b))))$
3	$i(h(x_{\mathbb{C}}, y_{\mathbb{B}}))$	$i(h(d, f(a, h(c, b), c))$	$i(h(i(h(c, b)), y))$

Problem 3.3 (Abstract Procedures)

6pt

6min

You are given the following abstract procedures over the ADT of unary natural numbers:

1. $\langle f{::}\mathbb{N} \times \mathbb{N} \to \mathbb{N}\,;\, \{f(n, o) \rightsquigarrow n, f(n, s(m)) \rightsquigarrow s(f(n, m))\}\rangle$
2. $\langle g{::}\mathbb{N} \times \mathbb{N} \to \mathbb{N}\,;\, \{g(n, o) \rightsquigarrow s(o), g(n, n) \rightsquigarrow s(o), g(s(n), s(m)) \rightsquigarrow f(g(n, s(m)), g(n, m))\}\rangle$

Write down the mathematical definition of each of these functions (e.g. $f(x) = x^2$ for the square function). What are their names? On what arguments does g return a meaningful result?

Solution:

1. addition.
2. combinations.

$g(n, m)$ gives a meaningful result whenever $n \geq m$.

4 Programming in Standard ML

Problem 4.1 (Chess mutual recursion)

10pt

8min

The Masters of chess have gotten "board" of playing simple chess all the time. They are inventing a 3D chess but for this they have to create a board. Since it is 3D they can't see inside of this new board and know if it is correctly painted or not. Write a pair of mutually recursive SML functions `black` and `white` that given a list of tuples of `(int * int * int) * bool` where the first one is the position of the cell $\langle x, y, z\rangle$ and the second one is `true` for black and `false` for white and returns `true` if it is a valid combination and `false` otherwise. Check that the input list has the right format, if not raise exceptions

Hint: Try to simplify the input to only have a `bool list` that models the chessboard in a way that you can call the mutually recursive functions

Solution:

```
fun black [] = true
  | black (a::b) = if not a then false else white(b)
and white [] = true
  | white (a::b) = if a then false else black(b)
```

Problem 4.2 (Atom energy levels)

Quantum physics has showed that every atom can only be found on a discrete array of states (levels). Each of these states is characterized by the energy that the atom possesses while on that level. It has been observed that each atom tries to reach the state with the lowest possible energy by doing successive transitions ("jumps") between states. This kind of transitions can be performed from any higher energy state to a lower one, not necessarily consecutive. Also, it is known that during a transition, the atom cannot lose more than half the energy it had before the "jump". 15pt 12min

Write a SML function level that takes as argument a list of integers, representing the energies of each level and returns the *number of all possible transitions* (through arbitrarily many intermediate states) between the highest and lowest energy state. It is known that initially the atom is found in the state with the highest energy. Also, you can assume the list of the energies is given in ascending order.

Note: The accuracy of the actual physical process described in this problem was partially neglected in order to provide a simpler coding task.

```
val level = fn : int list -> int

level ([10,15,25,29,45,76,90]) = 6;
```

The 6 possible transition chains are:

90 - 76 - 45 - 29 - 25 - 15 - 10
90 - 76 - 45 - 29 - 15 - 10
90 - 45 - 29 - 25 - 15 - 10
90 - 76 - 45 - 25 - 15 - 10
90 - 45 - 25 - 15 - 10
90 - 45 - 29 - 15 - 10

Solution:

```
fun find (n, []) = []
| find (n , x::xs) = if 2*x>=n andalso x<n
then x::find(n,xs)
else find(n,xs);

fun replace (l, []) = []
| replace (l, L::LS) = if L<>hd(l)
  then find(L,l) @ replace(l,LS)
else L::replace(l,LS);

fun check (n,[]) = false
| check (n,x::xs) = if n<>x then true
  else check(n,xs);
```

```
fun repeat(min,L,l) = if check(min,L)
        then repeat(min,replace(l,L),l)
        else List.length(L);

fun level(l) = let
     val min=hd(l)
    val L = hd(rev(l))::[]
in
     repeat(min,L,l)
end;
```