Name: Matriculation Number:

Midterm Exam
General CS 1 (320101)
October 25. 2005

You have one hour(sharp) for the test;
Write the solutions to the sheet.

The estimated time for solving this exam is 61 minutes, leaving you -1 minutes for revising your exam.

You can reach 61 points if you solve all problems. You will only need 58 points for a perfect score, i.e. 3 points are bonus points.

Different problems test different skills and knowledge, so do not get stuck on one problem.

	To be used for grading, do not write here									
prob.	1.1	1.2	1.3	1.4	2.1	3.1	3.2	4.1	Sum	grade
total	3	5	10	10	8	4	6	15	61	
reached										

Good luck to all students who take this test

1 Elementary Discrete Mathematics

3pt

Problem 1.1 (Greek Letters)
Fill in the blanks in the table of Greek letters. Note that capitalized names denote capital Greek letters.

Symbol	Σ	ρ	ξ	δ				
Name					*sigma*	*Phi*	*omega*	*psi*

Solution:

Symbol	Σ	ρ	ξ	δ	σ	Φ	ω	ψ
Name	*Sigma*	*rho*	*xi*	*delta*	*sigma*	*Phi*	*omega*	*psi*

Problem 1.2 (Function Definition) 5pt

Let A and B be sets. State the definition of the concept of a partial function with domain A and codomain B. Also state the definition of a total function with domain A and codomain B.

Solution: Let A and B be sets, then a relation $R \subseteq AB$ is called a **partial/total function**, iff for each $a \in A$, there is at most/exactly one $b \in B$, such that $\langle a, b \rangle \in R$.

Problem 1.3 **(Anti-Symmetry)** 10pt

We call a relation R **anti-reflexive** iff $\forall a \in A.\langle a, a\rangle \notin R$. and **anti-symmetric** iff $\forall a, b \in A.\langle a, b\rangle \in R \Rightarrow \langle b, a\rangle \notin R$.

Prove or refute that any anti-reflexive and transitive relation is also anti-symmetric.

Solution: Let R be an anti-reflexive and transitive relation on A. To show anti-symmetry, assume for a non-empty R that $\langle a, b\rangle \in R$ holds. Further more let $\langle b, a\rangle \in R$. Then we can infer from transitivity that $\langle a, a\rangle \in R$. Since R is anti-reflexive this is a contradiction and thus, $\langle b, a\rangle \notin R$. In case of a empty R this relation is trivially anti-symmetric.

Problem 1.4 (Induction) 10pt

Prove by induction or refute that for all natural numbers n the following assertion holds: $n^3 + 5n$ is divisible by 6.

Solution:

Assertion: $\forall n \in \mathbb{N}. \frac{n^3+5n}{6}$

Induction:

Base case: $n = 0$ $\qquad \frac{0^3+5\times 0}{6} = 0$

Step case: $n \neq 0$

Induction hypothesis $\qquad \forall n \in \mathbb{N}. \frac{n^3+5n}{6}$

Induction assertion $\qquad \forall n \in \mathbb{N}. \frac{n+1^3+5(n+1)}{6}$

$$\frac{n+1^3+5(n+1)}{6} = \frac{(n^2+2n+1^2\times(n+1))+5n+5}{6} =$$

$$\frac{n^3+n^2+2n^2+2n+n+1+5n+5}{6} = \frac{n^3+3n^2+8n+6}{6} =$$

$$\underbrace{\frac{n^3+5n}{6}}_{I.H.} + \frac{3n^2+3n+6}{6}$$

Now we have to prove that the second summand is also divisible by 6

$$\frac{3n^2+3n+6}{6} = \frac{3\times(n^2+n+2)}{6} = \frac{(n^2+n+2)}{2}$$

The last term is even, for the following reasons:
* The product/sum of even numbers is even.
* The product of odd numbers is odd, but the sum of two odd numbers is even. □

An alternative and more elegant solution:
Base cases:

1. $n = 0 : n^3 + 5n = 0$ which is divisible by 6
2. $n = 1 : n^3 + 5n = 6$ which is also divisible by 6

Step case from n to $n + 2$:

$$n + 2^3 + 5(n + 2) = (n^3 + 5n) + 6(n^2 + 2n + 1)$$

This term is divsible by 6 since the first summand is by the induction hypothesis and the second because of the factor 6.

2 Substitution

8pt

Problem 2.1 (Substitution Applications)
Let $\sigma := [(h(c))/x], [(g(a, f(a), b))/z]$ and $\tau := [a/x], [(h(b))/y], [c/z]$ be substitutions and $s := g(x, h(y), z)$ and $t := h(g(x, y, g(a, y, x)))$ constructor terms.

1. Give an abstract data type that makes these terms and substitutions well-sorted.
2. Give the 4 result terms of substitution application $\sigma(s)$, $\sigma(t)$, $\tau(s)$, and $\tau(t)$).

Solution: Simply take an abstract data type with only one sort S and constructor declarations where all constructors have S as domain (with pair sorts corresponding to their arity) and as codomain.

$$\begin{array}{rclrcl} \sigma(s) & = & g((h(c), h(y), g(a, f(a), b)) & \sigma(t) & = & h(g(h(c), y, g(a, y, h(c)))) \\ \tau(s) & = & g(a, h(h(b)), c) & \tau(t) & = & h(g(a, h(b), g(a, h(b), a))) \end{array}$$

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3 Abstract Data Types and Abstract Procedures

4pt

Problem 3.1 (SML datatypes vs Abstract Data Types)
Given the SML datatypes

1. `datatype A = a | f of A * A`
2. `datatype B = b | g of A -> B`

Write down one abstract data type in math notation representing both SML datatypes at once.

Solution:

$$\langle\{\mathbb{A},\mathbb{B}\},\{[a\colon\mathbb{A}],[b\colon\mathbb{B}],[f\colon\mathbb{A}\times\mathbb{A}\to\mathbb{A}],[g\colon\mathbb{A}\to\mathbb{B}\to\mathbb{B}]\}\rangle$$

Problem 3.2 (Ground Constructor Terms) 6pt

Assume a, b and f, g, h are constructors where a is of sort $\mathbb{A}$ and b of sort $\mathbb{B}$ with $\mathbb{A} \neq \mathbb{B}$.

1. Write down an appropriate abstract data type $\mathcal{A}$ such that $g(f(a,b),h(g(a,b)))$ is a ground constructor term in $\mathcal{A}$.
2. And for the same $\mathcal{A}$ you found justify whether or not $f(g(a,b),h(f(a,b)))$ is a ground constructor term in $\mathcal{A}$ too.

Solution:

1. $\langle\{\mathbb{A},\mathbb{B}\},\{[a\colon \mathbb{A}],[b\colon \mathbb{B}],[f\colon \mathbb{A}\times\mathbb{B}\to\mathbb{A}],[g\colon \mathbb{A}\times\mathbb{B}\to\mathbb{A}],[h\colon \mathbb{A}\to\mathbb{B}]\}\rangle$ (alternatively the domain of h and the codomain of g could be $\mathbb{B}$.)
2. $f(g(a,b),h(f(a,b)))$ is a ground constructor term in $\mathcal{A}$ too, since $f(a,b)$ and $g(a,b)$ have the same type. (If we had chosen $\mathbb{B}$ as domain of h and the codomain of g then the term $f(g(a,b),h(f(a,b)))$ were not a ground constructor term in $\mathcal{A}$.)

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4 Programming in Standard ML

15pt

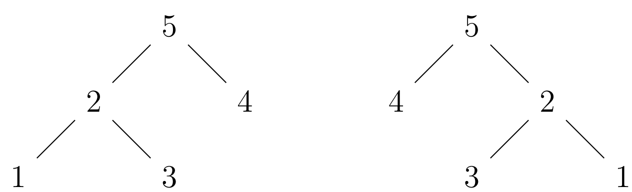
Problem 4.1 (Flip Binary Tree)

A *binary tree* is a relation T on $\mathbb{N}$, such that for every $n \in \mathbb{N}$ there are two or zero $m \in \mathbb{N}$, such that $\langle n, m \rangle \in T$, and exactly one $r \in \mathbb{N}$, such that there is no $p \in \mathbb{N}$ with $\langle p, r \rangle \in T$.

We can represent the set of binary trees as the abstract data type

$$\langle \{\mathbb{T}, \mathbb{N}\}, \{[leaf\colon \mathbb{N} \to \mathbb{T}], [branch\colon \mathbb{N} \times \mathbb{T} \times \mathbb{T} \to \mathbb{T}]\} \rangle$$

1. Provide an corresponding SML datatype declaration `btree`.
2. construct the binary trees below within SML



3. write an SML function that takes an binary tree and returns it flipped around its vertical axis, i.e. the function transforms the left tree into the right one and the other way around.

Solution:

1. `datatype btree = leaf of int | branch of int * btree * btree;`
2.
```
bintree1 = branch(5,(branch(2,(leaf 1),(leaf 3))),(leaf 4));
bintree2 = branch(5,(leaf 4),(branch(2,(leaf 3),(leaf 1))));
```

```
fun flip (branch (n,lb,rb)) = branch(n,flip(rb),flip(lb))
  | flip (leaf n) = (leaf n);
```
