Name: Matriculation Number:

Final Exam
General CS I (320101)

December 16, 2013

You have two hours(sharp) for the test;
Write the solutions to the sheet.

The estimated time for solving this exam is 0 minutes, leaving you 120 minutes for revising your exam.

You can reach 0 points if you solve all problems. You will only need 100 points for a perfect score, i.e. -100 points are bonus points.

Different problems test different skills and knowledge, so do not get stuck on one problem.

	To be used for grading, do not write here	
prob.	Sum	grade
total	0	
reached		

Please consider the following rules; otherwise you may lose points:

- "Prove or refute" means: If you think that the statement is correct, give a formal proof. If not, give a counter-example that makes it fail.
- Always justify your statements. Unless you are explicitly allowed to, do not just answer "yes" or "no", but instead prove your statement or refer to an appropriate definition or theorem from the lecture.
- If you write program code, give comments!

1 Mathematical Foundations

Problem 1.1 (Rabbits love carrots)

After the failed attempt to create a love relation in the second midterm, Dr. Flipidon decided that if he couldn't have love then no one could. He put on his glasses, messed up his hair and sat down to work. Knowing his task will be difficult, he decided to start with something simpler. He heard somewhere that rabbits love carrots more than anything else in the world, so he bought n rabbits. He knew that if he manages to create enough $c \times c$ matrices of carrots then the rabbits will lose the ability to love anything else than carrots. So he bought $(c+1)^n$ carrots. However, during the night the rabbits became loose and each one of them stole exactly c carrots. Petre, the greedy pink fluffy bunny, stole 1 more carrot and went to sleep. Will Dr. Flipidon be able to create a some of $c \times c$ matrices of carrots without leftovers (he does not want to reward the rabbits)? 10pt 15min

Prove by induction or refute that $(c+1)^n - cn - 1$ divisible by c^2 for all $n \geq 2$.

Hint: You can transform the formula in the step case to arrive a sum/difference of expressions, and argue about divisibility by c^2 for suitable combinations of them.

If you can not make this go through you may find an induction where the step case proceeds by steps of two easier, i.e. we go from n to $n+2$ (Think about the base case here).

Solution:

Proof: We prove by induction

P.1.1 Base case: $n = 2$:

P.1.1.1 $(c+1)^2 - 2c - 1 = c^2 + 2c + 1 - 2c - 1 = c^2$ □

P.1.2 Step case::

P.1.2.1 Assume that the condition is true for $n = k$.

P.1.2.2

$$\begin{aligned}
(c+1)^{k+1} - (k+1)c - 1 &= (c+1)^k(c+1) - kc - (c+1) \\
&= (c+1)((c+1)^k - 1) - kc \\
&= (c+1)(mc^2 + kc) - kc \\
&= mcc^2 + kc^2 + mc^2 + kc - kc \\
&= (mc + k + m)c^2
\end{aligned}$$

□

□

Another proof is shorter, but slightly more involved

Proof: We prove by induction with two base cases

P.1.1 Base case: $n = 2$:

P.1.1.1 $(c+1)^2 - 2c - 1 = (c^2 + 2c + 1) - 2c - 1 = c^2$ □

P.1.2 Base case: $n = 3$: $(c+1)^3 - 3c - 1 = (c^3 + 3c^2 + 3c + 1) - 3c - 1 = c^3 - 3c^2$ □

P.1.3 Step case::

P.1.3.1 Assume that the condition is true for n, we show it for $n+2$

P.1.3.2 Then we compute

$$\begin{aligned}
(c+1)^{n+2} - (n+2)c - 1 &= (c+1)^n(c+1)^2 - cn - 2c - 1 \\
&= 2c(c+1)^n + c^2(c+1)^n + (c+1)^n - cn - 1 - 2c \\
&= 2c(c+1)^n - 2c + c^2(c+1)^n + mc^2 \\
&= 2c((c+1)^n - 1) + c^2(c+1)^n + mc^2 \\
&= 2c((c+1)^n - cn - 1 + cn) + c^2(c+1)^n + mc^2 \\
&= 2c(kc^2 + cn) + c^2(c+1)^n + mc^2 \\
&= 2ckc^2 + 2c^2n + c^2(c+1)^n + mc^2
\end{aligned}$$

where m and k are factors that come out of the inductive hypothesis.

P.1.3.3 All summands are divisible by c^2, so the sum is as well □

□

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2 Abstract Data Types and Abstract Procedures

Problem 2.1 (ADT for people)

15pt
15min

After his successful experiment with the rabbits, Dr. Flipidon decided to start realizing his plan. But to do that he would have to know what he's dealing with in the first place.

Design and abstract data type for human beings, considering that each person is characterized by

- his/her sex (male or female),
- a name (string),
- a date of birth,
- eye-color (black, blue or green).
- soul of integer purity (unfortunately, can be negative as well!)
- requirements for a soulmate: the insist on a minimum purity of soul and a particular eye color to qualify.

As always, each person is looking for a soulmate. After writing an appropriate ADT to include and illustrate all of the above, design an abstract procedure which determines whether two people are soulmates. Of course, the sentiment has to be mutual.

Solution:

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3 Programming in Standard ML

Problem 3.1 (A spiral square)

25pt
20min

After studying human behavior, Dr. Flipidon learned that humans very often fall in love with someone at parties. In human parties, people always are on the dance floor forming a perfect square. The Doctor decided to use SML to help him figure in what order to ban people from experiencing love.

1. Write SML functions incr_seq and decr_seq that create lists of consecutive integers in increasing/decreasing order, given a starting number and length. Signature and example:

```
val decr_seq = fn : int * int -> int list
val incr_seq = fn : int * int -> int list

- incr_seq(3,4);
val it = [3,4,5,6] : int list
- decr_seq(11,5);
val it = [11,10,9,8,7] : int list
```

2. Write an SML function combine, that given a matrix and 2 columns appends the first one to the left of the matrix and the second one to the right of the matrix. The matrix is represented by an 'a list list (a list of its rows) and each of the 2 columns is represented by an 'a list. Signature and example:

```
val combine = fn : 'a list * 'a list list * 'a list -> 'a list list

- combine([1,2,3],[[4,4],[4,4],[4,4]],[1,2,3]);
val it = [[1,4,4,1],[2,4,4,2],[3,4,4,3]] : int list list
```

To represent the intution:

$$\mathsf{combine}\left([1,2,3],\ \begin{matrix}4 & 4\\ 4 & 4\\ 4 & 4\end{matrix},\ [1,2,3]\right) = \begin{matrix}1 & 4 & 4 & 1\\ 2 & 4 & 4 & 2\\ 3 & 4 & 4 & 3\end{matrix}$$

3. *Finally, Dr. Flipidon felt that it was best to select his victims from the outside parts of the dancefloor making his way to the center.* Write an SML function spiral, that given an integer n, creates a matrix of size $n \times n$ which contains the numbers from 1 to n^2 in a spiral order. Example for $n = 5$:

```
1  2  3  4  5
16 17 18 19 6
15 24 25 20 7
14 23 22 21 8
13 12 11 10 9
```

The matrix, of course, is represented by a list of its rows - int list list. Signature and example:

```
val spiral = fn : int -> int list list

- spiral(4);
val it = [[1,2,3,4],[12,13,14,5],[11,16,15,6],[10,9,8,7]] : int list list
```

This subtask is worth most of the points.

Hint: Think of how you can use the functions in the first two subtasks to solve the last one (even if you did not manage to write them). Consider a recursive solution where you construct a matrix of size n by adding another layer to a matrix of size $n-2$.

Solution:

1.
```
fun incr_seq(s,0) = []
  | incr_seq(s,n) = s::incr_seq(s+1,n-1);

fun decr_seq(s,0) = []
  | decr_seq(s,n) = s::decr_seq(s-1,n-1);
```

2.
```
fun combine(l::left, m::mid, r::right) = ([l]@m@[r])::combine(left, mid, right)
  | combine([],[],[]) = [];
```

3.
```
fun spiral_help(1,s) = [[s]]
  | spiral_help(2,s) = [[s,s+1],[s+3,s+2]]
  | spiral_help(n,s) =
      let
        val inner = spiral_help(n-2,n*4-4+s)
      in
        [incr_seq(s,n)] @
        combine(decr_seq(s+n*4-5,n-2),inner,incr_seq(s+n,n-2)) @
        [decr_seq(s+n*3-3,n)]
      end;

fun spiral(n) = spiral_help(n,1);
```

Problem 3.2 (Mutual Recursion)

15pt
10min

Write two mutually recursive SML functions odd and even that given $n > 0$ return how many words of length n over the alphabet $\{A, B, C, D\}$ there are which contain an odd number of Bs and an even number of Bs respectively. Make sure you raise appropriate exceptions. Signature and example:

```
val odd = fn : int -> int
val even = fn : int -> int

- odd(2);
val it = 6 : int
- even(3);
val it = 36 : int
```

Explanation for the first example: The words of length two with an odd number of Bs are BA, BC, BD, AB, CB and DB.

Solution:

```
exception nonpositive;

fun odd(1) = 1
  | odd(n) = if n<1 then raise nonpositive else 3*odd(n−1) + even(n−1)
and even(1) = 3
  | even(n) = if n<1 then raise nonpositive else 3*even(n−1) + odd(n−1);
```

4 Formal Languages and Codes

Problem 4.1 (Bragging in Code)

Happy about his recent discoveries, Dr. Flipidon decided to brag to his friends (the rabbits). He knows that rabbits only communicate in codes over the alphabet $\{C, B\}$ (carrots and rabbit babies) and knows that if you give them anything written in a prefix code they will always be able to decode it. 5pt 5min

Help Dr. Flipidon to come up with a prefix code with target alphabet $\{B, C\}$ and encrypt the message I DID IT! (note that you have to encode the blank/space as well).

Solution:

5 Boolean Algebra

Problem 5.1 (Boolean Expressions)

Now, it was finally time that Dr. Flipidon's plan went into action. Unfortunately for him, he lost some of the pills so he decided to go with a slightly different approach. He was to minimize the number of pills used, so he decided to come up with a formula that would tell him whether a certain combination of pills works. 8pt 10min

Given a formula for the effectiveness of the pills $e := x_1 * x_2 + \overline{(x_2 * x_3) * x_2 + x_3 * x_4}$ and a variable assignment $\varphi := [T/x_1], [F/x_2], [T/x_3], [F/x_4]$, compute $\mathcal{I}_\varphi(e)$ and give a (partial) trace of the computation.

We can read φ as a pills distribution, where $[T/x_i]$ means that pill x_i works and $\mathcal{I}_\varphi(e)$ to tell us whether the human will stop loving in this situation.

Solution:

Problem 5.2 (Boolean Expressions)

Dr. Flipidon found the following results when testing his pills: 10pt 15min

x_1	x_2	x_3	x_4	f
T	T	T	T	F
T	T	T	F	F
T	T	F	T	T
T	T	F	F	T
T	F	T	T	T
T	F	T	F	F
T	F	F	T	T
T	F	F	F	T
F	T	T	T	T
F	T	T	F	F
F	T	F	T	F
F	T	F	F	T
F	F	T	T	F
F	F	T	F	F
F	F	F	T	F
F	F	F	F	F

However, pills are expensive so he needs to minimize the total number of combinations he does with them (and subsequently minimize the number of pills). He heard of the Quine McCluskey algorithm and its widespread usage in GenCS.

Help him find a formula for the distribution of pills that minimizes the cost using this algorithm.

Solution:

6 Propositional Logic

Problem 6.1 (Hilbert calculus)

HIS PLAN SUCCEEDED!! Every person that attended that party now has lost the ability to love. After all that hard work, Dr. Flipidon as a good scientist had to draw conclusions from the experiment. He came to the conclusion that no one will love a person who doesn't love himself. Of course, he wanted to formalize that statement. The only way he knows how to formalize, though, is via the help of the Hilbert-style calculus $\mathcal{H}^0$. Recall that $\mathcal{H}^0$ has the two axioms 12pt 12min

1. $K := P \Rightarrow Q \Rightarrow P$
2. $S := (P \Rightarrow Q \Rightarrow R) \Rightarrow (P \Rightarrow Q) \Rightarrow P \Rightarrow R$

and the rules:

1. $\dfrac{\mathbf{A} \Rightarrow \mathbf{B} \quad \mathbf{A}}{\mathbf{B}}\,\text{MP}$

2. $\dfrac{\mathbf{A}}{[\mathbf{B}/X](\mathbf{A})}$ Subst

Prove that $P \Rightarrow Q \Rightarrow Q$ in $\mathcal{H}^0$.

Solution:
Proof:

P.1 **S** with $[P \Rightarrow P/Q]$ and $[P/R]$: $(P \Rightarrow (P \Rightarrow P) \Rightarrow P) \Rightarrow (P \Rightarrow P \Rightarrow P) \Rightarrow P \Rightarrow P$

P.2 **K** with $[P \Rightarrow P/Q]$: $P \Rightarrow (P \Rightarrow P) \Rightarrow P$

P.3 MP on **1.** and **2.**: $(P \Rightarrow P \Rightarrow P) \Rightarrow P \Rightarrow P$

P.4 **K** with $[P/Q]$: $P \Rightarrow P \Rightarrow P$

P.5 MP on **3.** and **4.**: $P \Rightarrow P$

P.6 **5.** with $[Q/P]$: $Q \Rightarrow Q$

P.7 **K** with $[Q \Rightarrow Q/P]$ and $[P/Q]$: $(Q \Rightarrow Q) \Rightarrow P \Rightarrow Q \Rightarrow Q$

P.8 MP on **7.** and **6.**: $P \Rightarrow Q \Rightarrow Q$ □

Problem 6.2 (Natural Deduction)

12pt

12min

Happy with the day's outcome, Dr. Flipidon finally decided to go to bed. While still not fully asleep another idea occured to him. If he, with the help of the pills, was able to stop everyone from loving, does that mean that without them he wouldn't be successful? Help him solve this dilemma.

Using $\mathcal{ND}^0$ calculus rules only prove $(A \Rightarrow B) \Rightarrow \neg B \Rightarrow \neg A$

Solution:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A \Rightarrow B]^1 \quad [A]^3}{B} \Rightarrow E \qquad [\neg B]^2}{F} FI}{\neg A} \neg I^3}{\neg B \Rightarrow \neg A} \Rightarrow I^2}{(A \Rightarrow B) \Rightarrow \neg B \Rightarrow \neg A} \Rightarrow I^1$$
