Name: Matriculation Number:

Final Exam
General CS 1 (320101)
December 13. 2005

You have two hours(sharp) for the test;
Write the solutions to the sheet.

The estimated time for solving this exam is 15 minutes, leaving you 105 minutes for revising your exam.

You can reach 107 points if you solve all problems. You will only need 100 points for a perfect score, i.e. 7 points are bonus points.

Different problems test different skills and knowledge, so do not get stuck on one problem.

	To be used for grading, do not write here										
prob.	1.1	1.2	2.1	3.1	3.2	4.1	5.1	6.1	7.1	Sum	grade
total	6	14	10	15	10	20	10	8	14	107	
reached											

Good luck to all students who take this test

- "Prove or refute" means: If you think that the statement is correct, give a formal proof. If not, give a counter-example that makes it fail.
- Always justify your statements. Unless you are explicitly allowed to, do not just answer "yes" or "no", but instead prove your statement or refer to an appropriate definition or theorem from the lecture.
- If you write program code, give comments!

1 Elementary Discrete Mathematics

6pt

Problem 1.1: Given $A := \{1,7,9,6\}$, $B := \{5,4,8\}$ and following relations:

$$R_1 \subseteq A \times A, \quad R_1 := \{\langle 7,9\rangle, \langle 9,7\rangle, \langle 1,1\rangle, \langle 1,6\rangle, \langle 6,1\rangle\}$$

$$R_2 \subseteq B \times B, \quad R_2 := \{\langle 8,4\rangle, \langle 5,5\rangle, \langle 4,4\rangle, \langle 8,8\rangle, \langle 8,5\rangle, \langle 5,4\rangle\}$$

Determine for these relations whether they are reflexive, symmetric, and transitive. If they are not, give counterexamples (i.e. examples, where the given property is violated).

Solution:

	reflexive	symmetric	transitive
R_1	N ($\langle 7,7\rangle \notin R_1$)	Y	N $\langle 7,9\rangle, \langle 9,7\rangle \in R_1$, but $\langle 7,7\rangle \notin R_1$
R_2	Y	N ($\langle 4,8\rangle \in R_1$, but $\langle 8,4\rangle \notin R_2$)	Y

Problem 1.2 (Converse Relations and Identity) 14pt

Let A and B sets, and $R \subseteq A \times B$ and $Q \subseteq B \times C$ relations. We define

- $Q \circ R := \{\langle a, c\rangle \mid \exists b \in B.\langle a, b\rangle \in R \wedge \langle b, c\rangle \in Q\}$
- $\mathrm{Id}_B := \{\langle b, b\rangle \mid b \in B\}$

Show that

1. there is a relation $P \subseteq A \times B$ such that $P \circ P^{-1} \not\subseteq \mathrm{Id}_B$
2. if f is a partial or total function from A to B then $f \circ f^{-1} \subseteq \mathrm{Id}_B$

Hint: Remember the definition of *converse*: $R^{-1} := \{\langle y, x\rangle \mid \langle x, y\rangle \in R\}$.

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2 Substitutions

10pt

Problem 2.1: Are there any terms A and B such that

1. $[A/y], [B/x](g(A, B)) = g(x, y)$
2. $[A/x](A) = [B/y](f(A, f(y, x)))$

is true? If so, name them. Explain your answer.

Solution:

3 Abstract Data Types and Abstract Procedures

15pt

Problem 3.1: Given the abstract data type of a binary tree

$$\langle\{\mathbb{T}\}, \{[leaf\colon \mathbb{T}], [branch\colon \mathbb{T} \times \mathbb{T} \to \mathbb{T}]\}\rangle$$

we define inductively two functions b and h on the set of binary trees:

- $b(t)$ returns the number of branches of the binary tree t:
  - $b(leaf) = 0$
  - $b(branch(h(lb, rb))) = 1 + b(lb) + b(rb)$
- $h(t)$ returns the height of the binary tree t:
  - $h(leaf) = 0$
  - $h(branch(lb, rb)) = 1 + \max(\{h(lb), h(rb)\})$

Prove by induction or refute on the structure of binary trees that we have $b(t) < 2^{(h(t))}$ for any binary tree t.

Solution:

Problem 3.2: Consider the following abstract procedure on the abstract data type of natural numbers: 10pt

$$\mathcal{P} := \langle f{::}\mathbb{N} \times \mathbb{N} \to \mathbb{N}\,;\ \{f(o, y_\mathbb{N}) \rightsquigarrow o, f(s(x_\mathbb{N}), y_\mathbb{N}) \rightsquigarrow y_\mathbb{N} + f(x_\mathbb{N}, y_\mathbb{N})\}\rangle$$

1. Show the computation process for $\mathcal{P}$ on the arguments $\langle s(s(o)), s(s(o))\rangle$.
2. Give the recursion relation of $\mathcal{P}$.
3. Does $\mathcal{P}$ terminate on all inputs? Justify your answer.
4. What function is computed by $\mathcal{P}$?

4 Programming in Standard ML

20pt

Problem 4.1 (Permutations)

1. Implement a function with the following type

   `intersperse: 'a -> 'a list -> 'a list list`

   The function takes a List `xs`, an element `y` and calculats all avaiable lists, whereas `y` has been inserted into `xs` at an abitrary position. Example:

   `intersperse 1 [2,3] = [[1,2,3],[2,1,3],[2,3,1]]`

2. Implement a function with the following type

   `permutations: 'a list -> 'a list list`

   that gives you all permutations of a list. Example:

   `permutations [1,2,3] = [[1,2,3],[2,1,3],[2,3,1],[1,3,2],[3,1,2],[3,2,1]]`

   Whether or not you successfully implemented the function `intersperse`, you may use it here.

Solution:

```
fun intersperse a [] = [[a]]
  | intersperse a (x::xs) = (a::x::xs) :: (map (fn z => (x::z)) (intersperse a xs));

fun permutations [] = [[]]
  | permutations (x::xs) =
    let
        fun mymap (h::tl) = (intersperse x h) @ mymap tl
          | mymap nil = nil
    in mymap (permutations xs) end;
```

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5 Complexity Analysis

10pt

Problem 5.1: Time complexity of an algorithm is stated as a function relating the input length to the number of evaluation steps in the worst case. Given the insort algorithm defined by the two SML functions

```
fun ins (x,[]) = [x]
  | ins (x,y::ys) = if x<=y then x::y::ys else y::ins(x,ys);

fun insort [] = []
  | insort (x::xs) = ins(x,insort xs);
```

Consider for simplicity only the comparison operation in `ins` as single evaluation step and disregard all other operations (e.g. concatenation).

1. how many evaluation steps will be taken to compute `insort [3,2,4,1]`?
2. what is the time complexity of insort in terms of Θ? Justify your answer!

Solution:

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6 Quine McClusky Algorithm

8pt

Problem 6.1: Use the algorithm of Quine-McCluskey and explain the intermediate steps to determine the minimal polynomial of the following function:

$x1$	$x2$	$x3$	$x4$	f
F	F	F	F	F
F	F	F	T	F
F	F	T	F	T
F	F	T	T	T
F	T	F	F	T
F	T	F	T	F
F	T	T	F	F
F	T	T	T	F
T	F	F	F	T
T	F	F	T	F
T	F	T	F	T
T	F	T	T	T
T	T	F	F	T
T	T	F	T	F
T	T	T	F	F
T	T	T	T	F

7 Tableau Calculus

14pt

Problem 7.1 (Refutation and model generation in Tableau Calculus)

1. Prove the following proposition

$$\models A \wedge (B \vee C) \Rightarrow (A \vee B) \wedge (A \vee C)$$

2. Find a model for following proposition

$$\models ((A \vee \neg C) \Rightarrow B) \wedge \neg (C \vee (B \Rightarrow \neg A))$$

Solution:

1. $$\begin{array}{c} A \wedge (B \vee C) \Rightarrow (A \vee B) \wedge (A \vee C)^{\mathsf{F}} \\ A \wedge (B \vee C)^{\mathsf{T}} \\ A^{\mathsf{T}} \\ B \vee C^{\mathsf{T}} \\ (A \vee B) \wedge (A \vee C)^{\mathsf{F}} \\ \begin{array}{c|c} A \vee B^{\mathsf{F}} & A \vee C^{\mathsf{F}} \\ A^{\mathsf{F}} & A^{\mathsf{F}} \\ B^{\mathsf{F}} & C^{\mathsf{F}} \\ \bot & \bot \end{array} \end{array}$$

2. $\varphi := \{A \mapsto \mathsf{T}, C \mapsto \mathsf{F}, B \mapsto \mathsf{F}\}$
