Final Exam
General CS 1 (320101)
December 13. 2005

NAME:

MATRICULATION NUMBER:

You have two hour (sharp) for the test;
Write the solutions to the sheet.

You can reach 107 points if you solve all problems. You will only need 100 points for a perfect score, i.e. 7 points are bonus points.

You have ample time, so take it slow and avoid rushing to mistakes!

Different problems test different skills and knowledge, so do not get stuck on one problem.

To be used for grading, do not write into this box											
prob.	1.1	1.2	2.1	3.1	3.2	4.1	5.1	6.1	7.1	Sum	grade
total	6	14	10	15	10	20	10	8	14	107	
reached											

1 Elementary Discrete Mathematics

Problem 1.1: Given $A := \{1, 7, 9, 6\}$, $B := \{5, 4, 8\}$ and following relations: 6pt

$$R_1 \subseteq A \times A, \quad R_1 := \{\langle 7, 9\rangle, \langle 9, 7\rangle, \langle 1, 1\rangle, \langle 1, 6\rangle, \langle 6, 1\rangle\}$$

$$R_2 \subseteq B \times B, \quad R_2 := \{\langle 8, 4\rangle, \langle 5, 5\rangle, \langle 4, 4\rangle, \langle 8, 8\rangle, \langle 8, 5\rangle, \langle 5, 4\rangle\}$$

Determine for these relations whether they are reflexive, symmetric, or transitive. If they are not, give counterexamples (i.e. examples, where the given property is violated).

Problem 1.2 (Converse Relations and Identity)
Let A and B sets, and $R \subseteq A \times B$ and $Q \subseteq B \times C$ relations. We define 14pt

- $Q \circ R := \{ \langle a, c\rangle \mid \exists b \in B. \langle a, b\rangle \in R \wedge \langle b, c\rangle \in Q \}$
- $\mathrm{Id}_B := \{ \langle b, b\rangle \mid b \in B \}$

Show that

1. there is a relation $P \subseteq A \times B$ such that $P \circ P^{-1} \not\subseteq \mathrm{Id}_B$
2. if f is a partial or total function from A to B then $f \circ f^{-1} \subseteq \mathrm{Id}_B$

Hint: Remember the definition of *converse*: $R^{-1} := \{ \langle y, x\rangle \mid \langle x, y\rangle \in R \}$.

2 Substitution

Problem 2.1: Are there any terms A and B such that 10pt

1. $[A/y], [B/x]g(A, B) = g(x, y)$
2. $[A/x]A = [B/y]f(A, f(y, x))$

is true? If so, name them. Explain your answer.

3 Abstract Data Types and Abstract Procedures

Problem 3.1: Given the abstract data type of a binary tree 15pt

$$\langle \{\mathbb{T}\}, \{[leaf : \mathbb{T}], [branch : \mathbb{T} \times \mathbb{T} \to \mathbb{T}]\}\rangle$$

we define inductively two functions b and h on the set of binary trees:

- $b(t)$ returns the number of branches of the binary tree t:
  - $b(leaf) = 0$

- $b(branch(lb, rb)) = 1 + b(lb) + b(rb)$

- $h(t)$ returns the height of the binary tree t:
  - $h(leaf) = 0$
  - $h(branch(lb, rb)) = 1 + max\{h(lb), h(rb)\}$

Prove by induction on the structure of binary trees that we have $b(t) < 2^{h(t)}$ for any binary tree t.

Problem 3.2: Consider the following abstract procedure on the abstract data type of natural numbers: 10pt

$$\mathcal{P} := \langle f :: \mathbb{N} \times \mathbb{N} \rightsquigarrow \mathbb{N}; \{f(o, y_\mathbb{N}) \rightsquigarrow o, f(s(x_\mathbb{N}), y_\mathbb{N}) \rightsquigarrow y_\mathbb{N} + f(x_\mathbb{N}, y_\mathbb{N})\}\rangle$$

1. Show the computation process for $\mathcal{P}$ on the arguments $\langle s(s(o)), s(s(o))\rangle$.
2. Give the recursion relation of $\mathcal{P}$.
3. Does $\mathcal{P}$ terminate on all inputs?
4. What function is computed by $\mathcal{P}$?

4 Programming in Standard ML

Problem 4.1: 20pt

1. Implement a function with the following type
   `intersperse: 'a -> 'a list -> 'a list list`

   The function takes a List `xs`, an element `y` and calculats all avaiable lists, whereas `y` has been inserted into `xs` at an abitrary position. Example:
   `intersperse 1 [2,3] = [[1,2,3],[2,1,3],[2,3,1]]`

2. Implement a function with the following type
   `permutations: 'a list -> 'a list list`

   that gives you all permutations of a list. Example:
   `permutations [1,2,3] = [[1,2,3],[2,1,3],[2,3,1],[1,3,2],[3,1,2],[3,2,1]]`

   Whether or not you successfully implemented the function `intersperse`, you may use it here.

5 Complexity Analysis

Problem 5.1: Time complexity of an algorithm is stated as a function relating the input length to the number of evaluation steps in the worst case. Given the insort algorithm defined by the two *SML* functions 10pt

```
fun ins (x,[]) = [x]
      | ins (x,y::ys) = if x<=y then x::y::ys else y::ins(x,ys);

    fun insort [] = []
      | insort (x::xs) = ins(x,insort xs);
```

Consider for simplicity only the comparison operation in `ins` as single evaluation step and disregard all other operations (e.g. concatenation).

1. how many evaluation steps will be taken to compute `insort [3,2,4,1]`?
2. what is the time complexity of insort in terms of Θ? Justify your answer!

6 Quine McClusky Algorithm

Problem 6.1: Use the algorithm of Quine McClusky and explain the intermediate steps to determine the minimal polynomial of the following function: 8pt

$x1$	$x2$	$x3$	$x4$	f
F	F	F	F	F
F	F	F	T	F
F	F	T	F	T
F	F	T	T	T
F	T	F	F	T
F	T	F	T	F
F	T	T	F	F
F	T	T	T	F
T	F	F	F	T
T	F	F	T	F
T	F	T	F	T
T	F	T	T	T
T	T	F	F	T
T	T	F	T	F
T	T	T	F	F
T	T	T	T	F

7 Tableau Calculus

Problem 7.1 (Refutation and model generation in Tableau Calculus)

14pt

1. Prove the following proposition

$$\models A \wedge (B \vee C) \Rightarrow (A \vee B) \wedge (A \vee C)$$

2. Find a model for following proposition

$$\models ((A \vee \neg C) \Rightarrow B) \wedge \neg (C \vee (B \Rightarrow \neg A))$$