The Intuitive Connection between Type and Set Theory

Florian Rabe

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The following correspondences provide the semantic intuition behind the syntax of type theory. Or, taking the opposite perspective: They show how type theory provides a formal concrete syntax for set theory. The set theoretic analogue of dependent types shows why we call $\lambda x{:}A.B$ or more generally anything of kind $\Pi x{:}A.\mathtt{type}$ a type family.

The first and lower second part of the table deal with simple and dependent types respectively. Within one part, the introduced symbols like s, t, A, and B are always typed/kinded the same way. The third part of the table shows how, optionally, simple ($A \times B$) and dependent ($\Sigma x{:}A.B$) products are added. (We have not talked about dependent products yet, and they are not part of LF. But they are the partner of dependent function types ($\Pi x{:}A.B$) and given here for completeness.)

Type Theory	Set Theory	
$A : \mathtt{type}$	$A \in \lvert\mathcal{S}\mathrm{et}\rvert$	
$s : A : \mathtt{type}$	$s \in A \in \lvert\mathcal{S}\mathrm{et}\rvert$	
$x : A \vdash t : B$	$t(x) \in B$	
$f = \lambda x{:}A.t \;:\; A \to B \;:\; \mathtt{type}$	$f : \begin{cases} A \to B \\ x \mapsto t(x) \end{cases}$	$f = \{(x, f(x)) \mid x \in A\} \subseteq A \times B$
$f\ s = t[s/x] : B$	$f(s) = t(s) \in B$	
$x : A \vdash B : \mathtt{type}$	$(B(x))_{x \in A}$ where $B(x) \in \lvert\mathcal{S}\mathrm{et}\rvert$ for all $x \in A$	
$C = \lambda x{:}A.B \;:\; \Pi x{:}A.\mathtt{type}$	$C : \begin{cases} A \to \lvert\mathcal{S}\mathrm{et}\rvert \\ x \mapsto B(x) \end{cases}$	
$x : A \vdash t : B$	$(t(x))_{x \in A}$ where $t(x) \in B(x)$ for all $x \in A$	
$f = \lambda x{:}A.t \;:\; \Pi x{:}A.B \;:\; \mathtt{type}$	$f : \begin{cases} A \to \bigcup_{x:A} B(x) \\ x \mapsto t(x) \in B(x) \end{cases}$	$f = \{(x, f(x)) \mid x \in A\} \subseteq A \times \bigcup_{x \in A} B$
$f\ s = t[s/x] : B[s/x]$	$f(s) = t(s) \in B(s)$	
if $s : A$ and $s' : B$		
$\langle s, s' \rangle \;:\; A \times B \;:\; \mathtt{type}$	$(s, s') \in A \times B$	
if $s : A$ and $s' : B[s/x]$		
$\langle s, s' \rangle \;:\; \Sigma x{:}A.B \;:\; \mathtt{type}$	$(s, s') \in \{(x, y) \mid x \in A, y \in B(x)\}$	