Monitoring the impact of teacher's intervention in inquiry-based mathematics learning with the use of dynamic geometry

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Abstract
In this study, we analyze computer-aided inquiry-based mathematics learning. A Moodle plug-in associated with the dynamic geometry software CindyJS which can record finegrained log data of learners' manipulations on the web was used. Our previous study indicates that teacher intervention can make student's inquiry systematic and exhaustive by helping them build a semantic circuit across language, symbolism, and visual images which are relevant to the targeted concept. In this study, we try to validate the impact of this kind of teacher intervention by monitoring the log data of manipulations.

Keywords
dynamic geometry, log of manipulation, productive failure, learning analytics

1. Introduction

For students to develop flexible scientific thinking, they should engage in solving problems which are complex and ill-structured. During problem solving, computer-based tools are often used to make students reflect on the data they have collected and speculate about the underlying mechanism [3]. While those tools enable students to decompose complex tasks and access key disciplinary content, they may prevent students from fully exploring the solution spaces and sufficiently evaluating alternative interpretations [4]. In fact, it has been shown that having students solve ill-structured problems without providing external support structure might endow their learning process in the longer term with hidden efficacy, even though the process is less efficient in the shorter term [5]. Therefore, close attention should be paid to the learning process so that students can make a full exploration of the concept they are studying while engaged in inquiry-based learning with computers. However, it is not easy to monitor learners' activities on computer because their thinking processes in inquiry tend to become highly complex. In fact, while several large-scale meta-analysis studies have shown that educational technology brought about significant improvements in mathematics achievement (for instance [1]), Cheung

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and Slavin [2] stated that the lack of information about the relationship between the process of technology use and the achievement measure might cause a discrepancy among similar studies in meta-analysis. Regarding the spread of mobile devices and high-speed internet connection, web-based systems should be preferred in inquiry-based mathematics learning. Moreover, high-resolution temporal data is needed in order to precisely analyze the temporal and sequential organization of the complex learning process [6]. Therefore, we have implemented a Moodle plug-in associated with the dynamic geometry software CindyJS (https://cindyjs.org) which is used to manipulate mathematical content dynamically. Using this plug-in, we can obtain the log data of learners' manipulations of CindyJS content on the web. In this study, we analyze the process of students' dynamically manipulating mathematics content to demonstrate that the teacher's preliminary intervention might guide their subsequent inquiry in a favorable direction.

2. Backgrounds and Research Questions

The topic analyzed in this research is the learning of polynomial approximation which is a typical theme in university level mathematics education. From the authors' experiences, it seems not to be so hard for the majority of students to apply the formula of Maclaurin's series

$$f(x) \approx f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

to specific functions. However, it is not so easy for them to accurately appreciate the associated concepts including the radius of convergence and the evaluation of the error terms. In fact, while the evaluation of the n-th error term

$$\varepsilon_n = f(x) - \left\{ f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n \right\}$$

expressed in the equation

$$\lim_{x \to 0} \frac{\varepsilon_n}{x^n} = 0$$

can be verified by combining the fundamental theorem of calculus and mathematical induction, the possibility that $f(x)$ cannot be approximated by any polynomial function globally can hardly be recognized by ordinary students only through paper-and-pencil-based learning. To observe those seemingly contradictory cases, inquiry-based learning with the use of computer-based tools is needed. For instance, the fact that $f(x)$ cannot be approximated by any polynomial unless $|x|$ is smaller than the radius of convergence can be observed when students use computer graphics tools and manipulate the graph of functions. Students are expected to fail fitting those graphs globally and find that fitting them is possible only in the neighbourhood of $x = 0$. The key point to be observed is that lower order terms are dominant in the neighbourhood of $x = 0$ while higher order terms are dominant in the region where $|x|$ is large. In order to ensure that students observe this point, it is necessary to monitor the activity of students and check whether their explorations are exhaustive or not.

In general, interpreting and scaffolding learners' mathematical thinking during their inquiry are not so easy because mathematics is a multi-semiotic activity whose resources are composed of many artifacts including gestures as a conceptual metaphor, written modes, spoken discourse, and visualization on digital media [7][8]. Moreover, mathematical concepts are constructed through the semantic circuit created by an interlocking network among the systems including language, symbolism, and visual images [9]. In the case of this research, the mathematical expressions

$$\lim_{x\to 0}\frac{x^m}{x^n}=0 \qquad \lim_{x\to\infty}\frac{x^m}{x^n}=\infty \qquad (m>n)$$

which show that the extent of convergence and divergence of monomial functions is determined by their orders should be understood while comparing the graphs of those functions. The result of our previous study indicates that the extent to which learners can build the above mentioned semantic circuit associated with the targeted concept might greatly influence the pattern of their manipulations relevant to the dynamic content [10]. Therefore, we can pose the following two research questions.

1. Is there any criterion derived from the log data of students' manipulating dynamic content to judge whether their explorations are sufficient or not?
2. Using this criterion, can we validate the effect of teachers' educational interventions to help students to extend the range of their explorations?

3. Methods

Due to the risk of COVID-19 infections, all classrooms were conducted fully online during the period of this research. For that reason, the authors implemented CindyJS content on the Moodle server and asked students to access that server and manipulate the dynamic content on the web. Figure 1 shows the Moodle page in which CindyJS content for this research was implemented.

Using this content, students were asked to find the best approximation near $x=0$

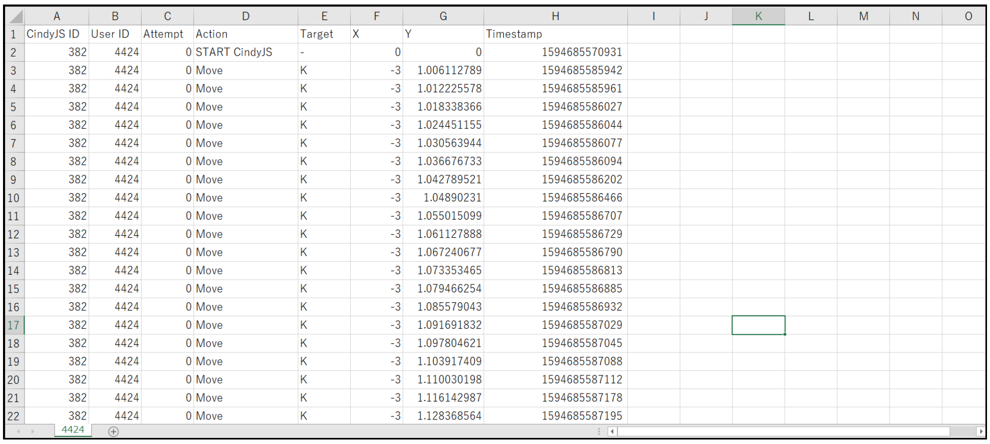
$$\sqrt{1+x}\approx a+bx+cx^2+d^3$$

of the target function $y=\sqrt{1+x}$ with cubic polynomial function by manipulating four sliders in the content. When the coefficients a,b,c,d are changed by moving the red points, the resulting graph of the cubic polynomial function is modified correspondingly. Thus, the task is to find the coefficients with which this graph (red curve) fits well to the graph of the function $y=\sqrt{1+x}$ (blue curve) near the point $(0,1)$. The log data of students' manipulations are stored on the Moodle server and are formatted into a csv file as in Figure 2.

On the one hand, Figure 3 shows the locally optimal approximation derived from the formula of Maclaurin's series. Unless $y=a+bx$ is set to be the tangent line at $(0,1)$, any choice of higher degree coefficients does not provide a suitable local approximation since cx^2 and dx^3 are the infinitesimals of higher order compared to the first order terms $a+bx$.

On the other hand, Figure 4 shows the globally optimal approximation with respect to the L^2 norm on the whole interval $[-1,2]$. Setting the first order part to be the equation of the

Figure 1: CindyJS content implemented on Moodle



	A	B	C	D	E	F	G	H
1	CindyJS ID	User ID	Attempt	Action	Target	X	Y	Timestamp
2	382	4424	0	START CindyJS	-	0	0	1594685570931
3	382	4424	0	Move	K	-3	1.006112789	1594685585942
4	382	4424	0	Move	K	-3	1.012225578	1594685585961
5	382	4424	0	Move	K	-3	1.018338366	1594685586027
6	382	4424	0	Move	K	-3	1.024451155	1594685586044
7	382	4424	0	Move	K	-3	1.030563944	1594685586077
8	382	4424	0	Move	K	-3	1.036676733	1594685586094
9	382	4424	0	Move	K	-3	1.042789521	1594685586202
10	382	4424	0	Move	K	-3	1.04890231	1594685586466
11	382	4424	0	Move	K	-3	1.055015099	1594685586707
12	382	4424	0	Move	K	-3	1.061127888	1594685586729
13	382	4424	0	Move	K	-3	1.067240677	1594685586790
14	382	4424	0	Move	K	-3	1.073353465	1594685586813
15	382	4424	0	Move	K	-3	1.079466254	1594685586885
16	382	4424	0	Move	K	-3	1.085579043	1594685586932
17	382	4424	0	Move	K	-3	1.091691832	1594685587029
18	382	4424	0	Move	K	-3	1.097804621	1594685587045
19	382	4424	0	Move	K	-3	1.103917409	1594685587088
20	382	4424	0	Move	K	-3	1.110030198	1594685587112
21	382	4424	0	Move	K	-3	1.116142987	1594685587178
22	382	4424	0	Move	K	-3	1.128368564	1594685587195

Figure 2: Log data of students' manipulations

tangent line causes difficulty in finding a suitable global approximation. Through observing this trade-off relation, learners are expected to appreciate intuitively the concept of the degree of the infinitesimal and the radius of convergence.

As seen in these figures, it is not easy to estimate the range in which the target function can be locally approximated by using a cubic polynomial function. In this sense, this task is complex and ill-structured. While the use of dynamic geometry is expected to play a crucial role, there is some risk that students will search the approximation without considering the power balance between monomials. To avoid this risk, teachers should turn students' close attention to the order of monomials with which their power balance and convergence range are correlated. Figure 5 is a screenshot of a supplementary video prepared for this educational intervention. In this video, it is explained that the graph of cubic function is plotted by superposing each monomial functions and that the monomial function x^n of higher degree n is a major factor in the region $|x| \gg 0$ whereas it is a minor factor in the region $x \sim 0$. These points were explained

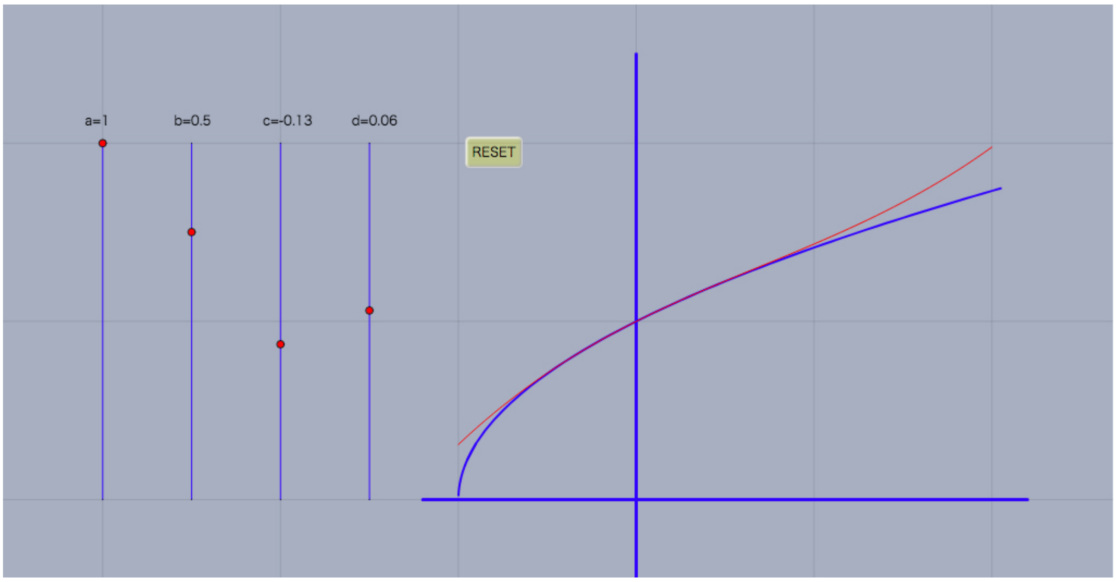


Figure 3: Locally optimal approximation

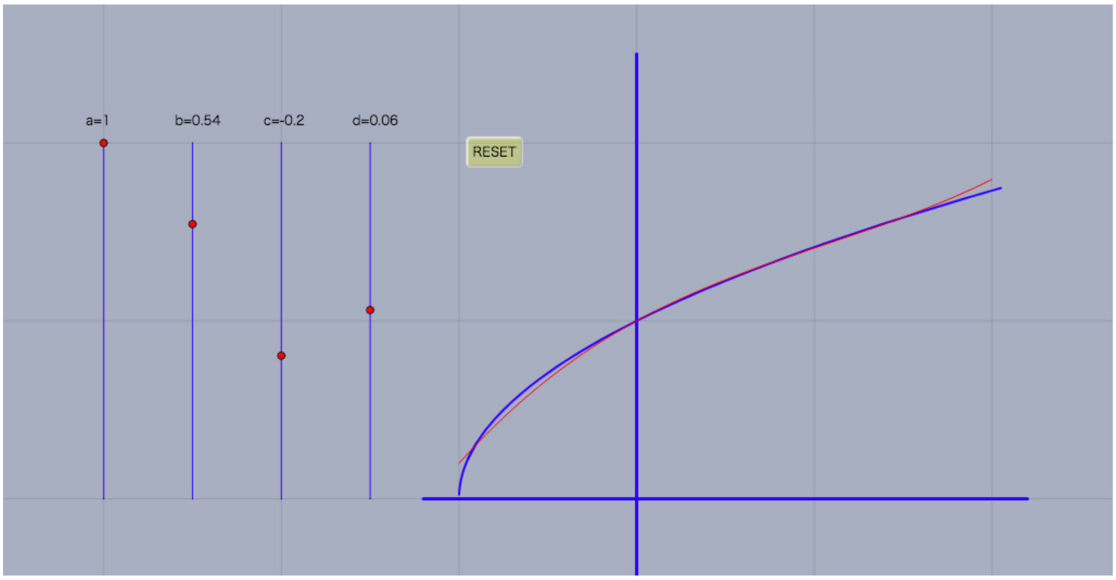


Figure 4: Globally optimal approximation

by showing both the symbolic representation of reduction with lower order monomial and visual images of superposing monomial functions.

Subjects in this study were first grade students in a Japanese university. Prior to the lesson used for this research, they had been given some elementary lectures concerning polynomial approximation and Macraulin's series of functions. While the evaluation of error terms to-

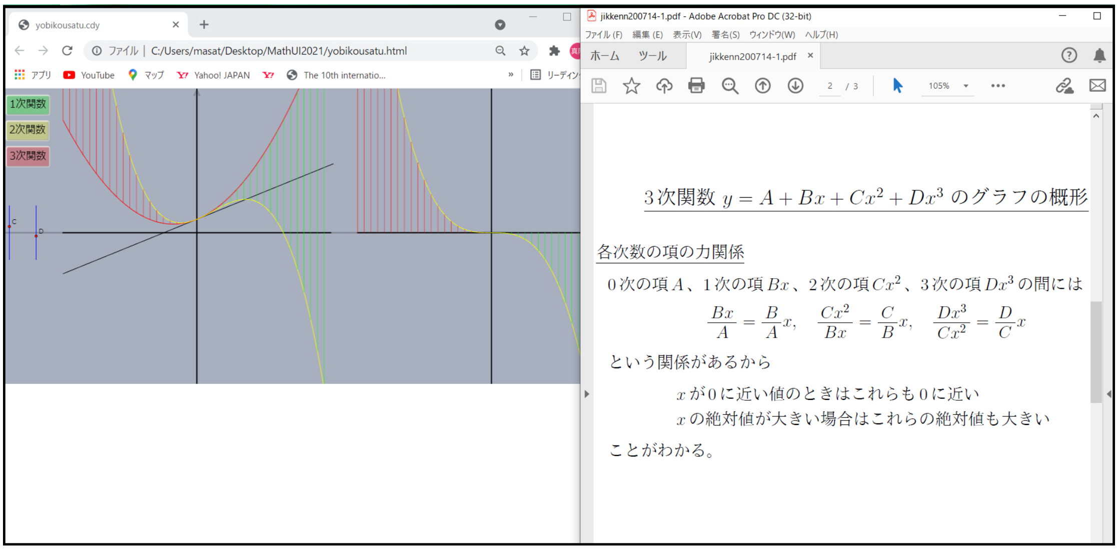


Figure 5: Supplementary video

gether with the formula to calculate the approximation had been taught by using mathematical expressions, no graphical explanations had been given. Students were randomly assigned to experimental group E (15 males and 36 females) and control group C (19 males and 34 females). Before they began manipulating the content, the video in Figure 5 was shown to group E and another video including brief instruction of the usage of the content was shown to group C.

4. Results

Since the region where learners try to fit one graph to another can be monitored by watching the maximal gap between two graphs on the relevant regions, we used the log data derived from the Moodle plug-in to compute the maximal gaps on the three regions $[-1, -0.5]$, $[-0.5, 0.5]$, $[0.5, 2]$ and graphed their temporal transition in black, red, and blue respectively. A sample graph is shown in Figure 6 where the horizontal axis represents the passage of time and the vertical one represents the maximal gap. The former has marks for every 50 seconds and the latter has marks for every $\frac{1}{6}$.

Since the lower order terms are dominant in the region $x \sim 0$, the low value of maximal gap on $[-0.5, 0.5]$ (red curve) shows that the first order part is set to be near to the equation of tangential line. This means that the learner attached some importance to the local approximation in the region $x \sim 0$ at that moment. On the contrary, in the case when the value of maximal gap on $[-0.5, 0.5]$ is high, that on $[-1, -0.5]$ (black curve) or that on $[0.5, 2]$ (blue curve) often falls. This means that the learner aimed at the approximation in the region apart from $x = 0$ at that moment. If we emphasize the effectiveness of manipulation, the case shown in Figure 7 is ideal. In fact, the red curve falls quickly and the final result ($a = 1,\ b = 0.5,\ c = -0.10,\ d = 0.04$) is very near to that obtained by using the formula of Maclaurin's series. However, the main

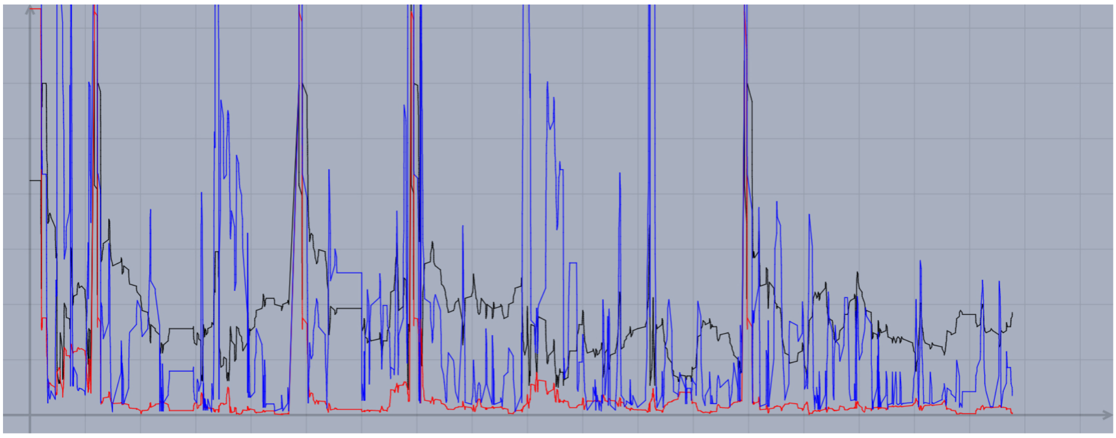


Figure 6: Visualization of the maximal gaps

Figure 7: Seemingly desirable manipulation process

purpose of this trial is to let learners observe as many cases as possible and the short duration of the manipulation process indicates that it is uncertain whether the learner explored exhaustive cases to recognize the trade-off relation mentioned above.

In the case when learners could recognize the necessity of setting the first order part to be the equation of the tangential line by manipulating higher order coefficients with various values of lower order coefficients, one further trial is needed in which they minimize the gap on $[-1, -0.5]$ while keeping the gap on $[-0.5, 0.5]$ small. Figure 8 shows a sample case of those trials. Because of the singularity of the derivative function of $y = \sqrt{1+x}$ at the point $(-1, 0)$, learners are expected to encounter the difficulty in fitting two graphs near that point.

Contrarily, the fitting on the region $[0, 2]$ is not so difficult as shown in Figure 9. These observations should lead to the understanding of the radius of convergence.

In that sense, the inquiry shown in Figure 7 is insufficient. In fact, though the black curve falls once, the red curve rose at that time. This indicates the possibility that the learner did not

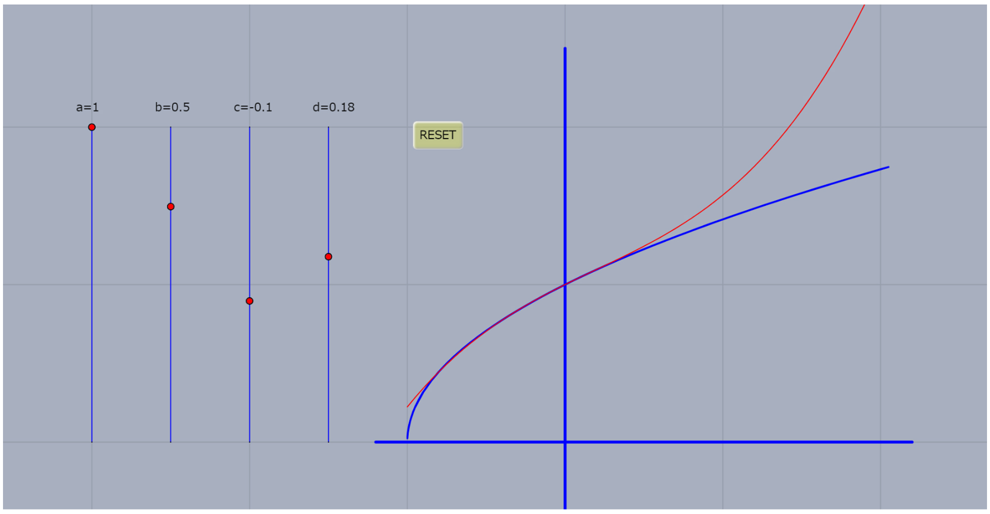


Figure 8: Some further manipulation (I)

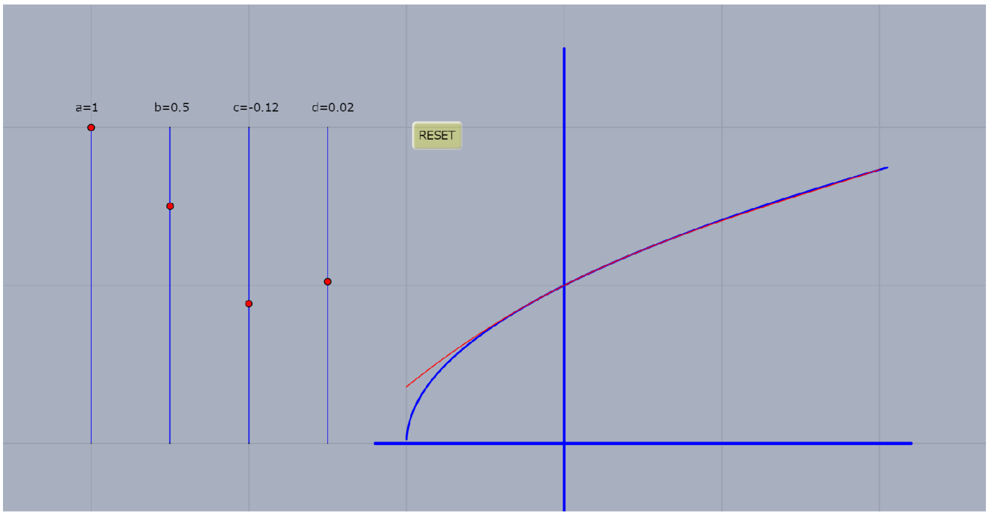


Figure 9: Some further manipulation (II)

recognize the power balance between monomial functions and his/her manipulation strategy depended on contingency. Surface observation of graphs for all participants suggested that learners in group E observed the situation as in Figure 8 more often than those in group C.

Based on this consideration, we adopted the criteria which is given by simultaneously using the maximal gaps on $[-1, -0.5]$ and $[-0.5, 0.5]$. Specifically, we counted the number of students whose manipulation process included the situation in which the maximal gap on $[-1, -0.5]$ attained the value smaller than the prescribed thresholds $0.20, 0.25, 0.30, 0.35, 0.40$ while the maximal gap on $[-0.5, 0.5]$ was smaller than 0.01. Here, the threshold 0.01 for the maximal gap on $[-0.5, 0.5]$ was chosen since the maximal gap between the target function $\sqrt{1+x}$ and its

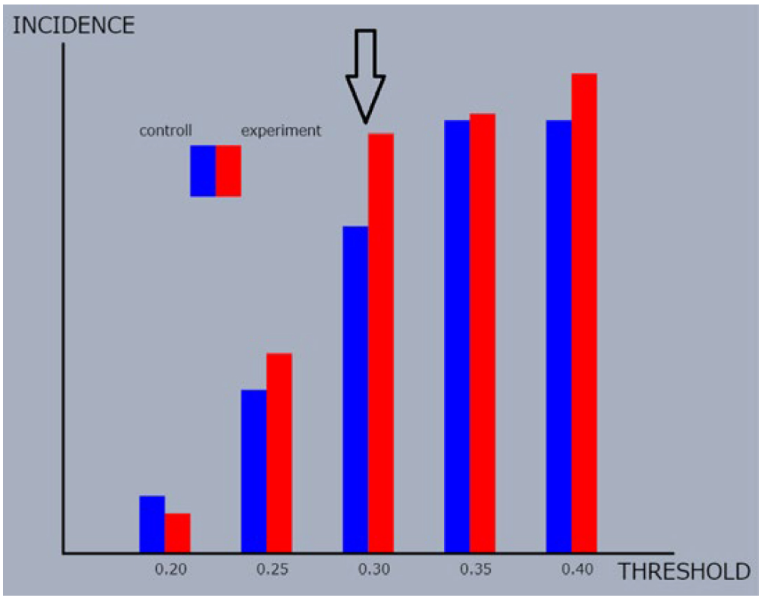


Figure 10: Incidences corresponding to several thresholds

second order approximation $1 + \frac{1}{2}x - \frac{1}{8}x^2$ on $[-0.5, 0.5]$ is very near to it. Figure 10 shows the ratio of these students among each group for each of the above thresholds of the maximal gap on $[-1, -0.5]$. Here the results of group E and C are represented in red and blue respectively.

Whereas no significant differences in the ratio are identified for the thresholds other than 0.30, the ratio for group E (42/51) is significantly higher than that for group C (34/53) in case of the threshold 0.30 as shown by the arrow in Figure 10. In fact, the p-value generated from"prop.test" function of R applied to this data is 0.01822. Regarding the fact that no advice about the manipulation strategy was given in the video, this result strongly suggests that the teachers' intervention through supplementary video induced learners' additional exploration in which they can observe the difficulty in making compatible choice of the approximation on $[-1, -0.5]$ and that on $[0, 2]$.

5. Discussion and Future Work

It can be seen that the above mentioned difference in the pattern of manipulation process between group E and C was caused by the following mechanism.

1. The learners in group C should have understood the graph shape of cubic function globally through the usual drawing procedure using derivative sign chart. Therefore, some of them might observe that the maximal gap on the region $[1, 2]$ became fairly large when they increased the value of d on the way to the situation in Figure 8 and stopped further manipulation in that direction.
2. Since the learners in group E watched the supplementary video, they should have recognized that the third order term dx^3 does not have a major influence on the graph shape

of cubic function in the region $x \sim 0$. Therefore, many of them may have recognized the possibility that the increase of the value of d can reduce the maximal gap on the region $[-1, 0]$ and therefore tried the case in Figure 8.

In summary, teacher's intervention using a supplementary video can be seen to have helped students build the interlocking network among the systems including language, symbolism, and visual images which made their inquiry more systematic and more exhaustive. In this sense, the result of this research indicates that the temporal transition of learners' thinking during their inquiry with the use of dynamic content is reflected in their manipulation process and is preceded by the activities based on these resources. Therefore, the mathematical user interface equipped with the system to store the log data of learners' manipulating dynamic content is indispensable for monitoring learners' inquiry and giving appropriate advice to them.

While the "productive failure" which the subjects experienced through their "extra" trial as mentioned above helped them appreciate correctly the target concept, that failure can make their manipulation process complex and divergent. This is because learners change perspectives over the course of extended experiences for solving ill-structured problems. The complexity and divergence of the learners' manipulation process make it very hard for ordinary teachers to make sense of and give support to learners' thinking. The result of this study strongly indicates that monitoring the appropriate signals derived from the log data of learners' manipulation process might enable ordinary teachers to infer learners' thinking and find appropriate ways of intervening.

In this study, there are many points to be improved. Though in this study we could diagnose the process of learners' mathematical inquiry by using the criteria based on the information derived from several moments in the whole process, it is necessary to make full use of the information relating to time and order in general. Moreover, teacher's intervention is usually carried out during the learners' inquiry and while monitoring their activities, whereas in this study it was carried out by using a video which learners watched before their inquiry. Thus, in order to make the workflow of this study applicable to a more realistic situation, it is necessary to develop a more advanced system to analyze the log data and visualize the result of that analysis in the instant of learners' inquiry.

Moreover, some CSCL (Computer-Supported Collaborative Learning) research investigating the causal relationship between discourse, manipulation, and gesture is needed to make the interpretation of log data (or the plausible choice of signal) grounded in the light of educational purpose. In our pilot study, a CSCL environment was prepared as seen in Figure 11. The result of this pilot study indicated that discourse and gesture are strongly correlated to the strategy of manipulating dynamic content and they can give some evidence for the interpretation of log data. While several methodologies for analyzing mathematical cognition are proposed, those based on the data derived from learners' linguistic activities and body movements seem to be most reliable at this stage.

Acknowledgments

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Figure 11: Mathematical inquiry using dynamic content in CSCL environment

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