Artificial Intelligence 1
Winter Semester 2024/25

– Lecture Notes –
Part I: Getting Started with AI

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Enough philosophy about "Intelligence" (Artificial or Natural)

- So far we had a nice philosophical chat, about "intelligence" et al.
- As of today, we look at technical stuff!

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- As of today, we look at technical stuff!
- Before we go into the algorithms and data structures proper, we will
  1. introduce a programming language for AI-1
  2. prepare a conceptual framework in which we can think about "intelligence" (natural and artificial), and
  3. recap some methods and results from theoretical computer science.

Chapter 3
Logic Programming

3.1 Introduction to Logic Programming and ProLog

Logic Programming

- **Idea:** Use logic as a programming language!
- We state what we know about a problem (the program) and then ask for results (what the program would compute).
- **Example 1.1.**

Program	Leibniz is human Sokrates is human Sokrates is a greek Every human is fallible	$x + 0 = x$ If $x + y = z$ then $x + s(y) = s(z)$ 3 is prime
Query	Are there fallible greeks?	is there a z with $s(s(0)) + s(0) = z$
Answer	Yes, Sokrates!	yes $s(s(s(0)))$

- **How to achieve this?** Restrict a logic calculus sufficiently that it can be used as computational procedure.
- **Remark:** This idea leads a totally new programming paradigm: logic programming.
- **Slogan:** Computation = Logic + Control (Robert Kowalski 1973; [Kow97])
- We will use the programming language Prolog as an example.

Prolog Terms and Literals

- **Definition 1.2.** Prolog expresses knowledge about the world via
  - constants denoted by lowercase strings,
  - variables denoted by strings starting with an uppercase letter or _, and
  - functions and predicates (lowercase strings) applied to terms.
- **Definition 1.3.** A Prolog term is
  - a Prolog variable, or constant, or
  - a Prolog function applied to terms.

  A Prolog literal is a constant or a predicate applied to terms.
- **Example 1.4.** The following are
  - Prolog terms: john, X, _, father(john), ...
  - Prolog literals: loves(john,mary), loves(john,_), loves(john,wife_of(john)),...

Prolog Programs: Facts and Rules

- **Definition 1.5.** A Prolog program is a sequence of clauses, i.e.
  - facts of the form l., where l is a literal, (a literal and a dot)
  - rules of the form h:$-b_1$,. . .,b_n., where $n > 0$. h is called the head literal (or simply head) and the b_i are together called the body of the rule.

  A rule h:$-b_1$,. . .,b_n., should be read as *h (is true) if b_1 and . . . and b_n are*.
- **Example 1.6.** Write "something is a car if it has a motor and four wheels" as car(X) :− has_motor(X),has_wheels(X,4). (variables are uppercase)
  This is just an ASCII notation for $m(x) \wedge w(x,4) \Rightarrow car(x)$.
- **Example 1.7.** The following is a Prolog program:

```
human(leibniz).
human(sokrates).
greek(sokrates).
fallible(X):−human(X).
```

The first three lines are Prolog facts and the last a rule.

Prolog Programs: Knowledge bases

- **Intuition:** The knowledge base given by a Prolog program is the set of facts that can be derived from it under the if/and reading above.
- **Definition 1.8.** The knowledge base given by Prolog program is that set of facts that can be derived from it by Modus Ponens (MP), $\wedge I$ and instantiation.

$$\frac{A \quad A \Rightarrow B}{B}\,\mathrm{MP} \qquad \frac{A \quad B}{A \wedge B}\,\wedge I \qquad \frac{\mathsf{A}}{[\mathsf{B}/X](\mathsf{A})}\,\mathrm{Subst}$$

Querying the Knowledge Base: Size Matters

- **Idea:** We want to see whether a fact is in the knowledge base.
- **Definition 1.9.** A query is a list of Prolog literals called goal literal (also subgoals or simply goals). We write a query as $?{-}A_1, \ldots, A_n.$ where A_i are goals.
- **Problem:** Knowledge bases can be big and even infinite. (cannot pre-compute)
- **Example 1.10.** The knowledge base induced by the Prolog program

```
nat(zero).
nat(s(X)) :− nat(X).
```

contains the facts nat(zero), nat(s(zero)), nat(s(s(zero))), ...

Querying the Knowledge Base: Backchaining

- **Definition 1.11.** Given a query Q: $?- A_1,\ldots,A_n$. and rule R: $h{:-}\ b_1,\ldots,b_n$, backchaining computes a new query by
  1. finding terms for all variables in h to make h and A_1 equal and
  2. replacing A_1 in Q with the body literals of R, where all variables are suitably replaced.
- Backchaining motivates the names goal/subgoal:
  - the literals in the query are "goals" that have to be satisfied,
  - backchaining does that by replacing them by new "goals".
- **Definition 1.12.** The Prolog interpreter keeps backchaining from the top to the bottom of the program until the query
  - succeeds, i.e. contains no more goals, or (answer: **true**)
  - fails, i.e. backchaining becomes impossible. (answer: false)
- **Example 1.13 (Backchaining).** We continue **??**

```
?- nat(s(s(zero))).
?- nat(s(zero)).
?- nat(zero).
true
```

Querying the Knowledge Base: Failure

- If no instance of a query can be derived from the knowledge base, then the Prolog interpreter reports failure.
- **Example 1.14.** We vary **??** using 0 instead of zero.

```
?- nat(s(s(0))).
?- nat(s(0)).
?- nat(0).
FAIL
false
```

Querying the Knowledge base: Answer Substitutions

- **Definition 1.15.** If a query contains variables, then Prolog will return an answer substitution as the result to the query, i.e the values for all the query variables accumulated during repeated backchaining.
- **Example 1.16.** We talk about (Bavarian) cars for a change, and use a query with a variables

```
has_wheels(mybmw,4).
has_motor(mybmw).
car(X):−has_wheels(X,4),has_motor(X).
?− car(Y) % query
?− has_wheels(Y,4),has_motor(Y). % substitution X = Y
?− has_motor(mybmw). % substitution Y = mybmw
Y = mybmw % answer substitution
true
```

PROLOG: Are there Fallible Greeks?

- **Program:**

```
human(leibniz).
human(sokrates).
greek(sokrates).
fallible(X):−human(X).
```

- **Example 1.17 (Query).** ?−fallible(X),greek(X).
- **Answer substitution:** $[\mathrm{sokrates}/X]$

3.2 Programming as Search

3.2.1 Knowledge Bases and Backtracking

Depth-First Search with Backtracking

- So far, all the examples led to direct success or to failure. (simple KB)
- **Definition 2.1 (Prolog Search Procedure).** The Prolog interpreter employs top-down, left-right depth first search, concretely, Prolog search:
  - works on the subgoals in left right order.
  - matches first query with the head literals of the clauses in the program in top-down order.
  - if there are no matches, fail and backtracks to the (chronologically) last backtrack point.
  - otherwise backchain on the first match, keep the other matches in mind for backtracking via backtrack points.

  We say that a goal G matches a head H, iff we can make them equal by replacing variables in H with terms.
- We can force backtracking to compute more answers by typing ;.

Backtracking by Example

- **Example 2.2.** We extend **??**:

```
has_wheels(mytricycle,3).
has_wheels(myrollerblade,3).
has_wheels(mybmw,4).
has_motor(mybmw).
car(X):-has_wheels(X,3),has_motor(X). % cars sometimes have three wheels
car(X):-has_wheels(X,4),has_motor(X). % and sometimes four.
?- car(Y).
?- has_wheels(Y,3),has_motor(Y). % backtrack point 1
Y = mytricycle % backtrack point 2
?- has_motor(mytricycle).
FAIL % fails, backtrack to 2
Y = myrollerblade % backtrack point 2
?- has_motor(myrollerblade).
FAIL % fails, backtrack to 1
?- has_wheels(Y,4),has_motor(Y).
Y = mybmw
?- has_motor(mybmw).
Y=mybmw
true
```

3.2.2 Programming Features

Can We Use This For Programming?

- **Question:** What about functions? E.g. the addition function?
- **Question:** We cannot define functions, in Prolog!
- **Idea (back to math):** use a three-place predicate.
- **Example 2.3.** add(X,Y,Z) stands for X+Y=Z
- Now we can directly write the recursive equations $X + 0 = X$ (base case) and $X + s(Y) = s(X + Y)$ into the knowledge base.

```
add(X,zero,X).
add(X,s(Y),s(Z)) :- add(X,Y,Z).
```

- Similarly with multiplication and exponentiation.

```
mult(X,zero,zero).
mult(X,s(Y),Z) :- mult(X,Y,W), add(X,W,Z).

expt(X,zero,s(zero)).
expt(X,s(Y),Z) :- expt(X,Y,W), mult(X,W,Z).
```

More Examples from elementary Arithmetic

- **Example 2.4.** We can also use the add relation for subtraction without changing the implementation. We just use variables in the "input positions" and ground terms in the other two. (possibly very inefficient "generate and test approach")

```
?−add(s(zero),X,s(s(s(zero)))).
X = s(s(zero))
true
```

- **Example 2.5.** Computing the n^{th} Fibonacci number (0, 1, 1, 2, 3, 5, 8, 13,...; add the last two to get the next), using the addition predicate above.

```
fib(zero,zero).
fib(s(zero),s(zero)).
fib(s(s(X)),Y):−fib(s(X),Z),fib(X,W),add(Z,W,Y).
```

- **Example 2.6.** Using Prolog's internal floating-point arithmetic: a goal of the form ?− D **is**e. — where e is a ground arithmetic expression binds D to the result of evaluating e.

```
fib(0,0).
fib(1,1).
fib(X,Y):− D is X − 1, E is X − 2,fib(D,Z),fib(E,W), Y is Z + W.
```

Adding Lists to Prolog

- **Definition 2.7.** In Prolog, lists are represented by list terms of the form
  1. [a,b,c,...] for list literals, and
  2. a first/rest constructor that represents a list with head F and rest list R as [F|R].
- **Observation:** Just as in functional programming, we can define list operations by recursion, only that we program with relations instead of with functions.
- **Example 2.8.** Predicates for member, append and reverse of lists in default Prolog representation.

```
member(X,[X|_]).
member(X,[_|R]):−member(X,R).

append([],L,L).
append([X|R],L,[X|S]):−append(R,L,S).

reverse([],[]).
reverse([X|R],L):−reverse(R,S),append(S,[X],L).
```

Relational Programming Techniques

- **Example 2.9.** Parameters have no unique direction "in" or "out"

```
?- rev(L,[1,2,3]).
?- rev([1,2,3],L1).
?- rev([1|X],[2|Y]).
```

- **Example 2.10.** Symbolic programming by structural induction:

```
rev([],[]).
rev([X|Xs],Ys) :- ...
```

- **Example 2.11.** Generate and test:

```
sort(Xs,Ys) :- perm(Xs,Ys), ordered(Ys).
```

3.2.3 Advanced Relational Programming

Specifying Control in Prolog

- *Remark 2.12.* The running time of the program from **??** is not $\mathcal{O}(n\log_2(n))$ which is optimal for sorting algorithms.

  ```
  sort(Xs,Ys) :- perm(Xs,Ys), ordered(Ys).
  ```

- **Idea:** Gain computational efficiency by shaping the search!

Functions and Predicates in Prolog

- *Remark 2.13.* Functions and predicates have radically different roles in Prolog.
  - Functions are used to represent data. (e.g. father(john) or s(s(zero)))
  - Predicates are used for stating properties about and computing with data.
- *Remark 2.14.* In functional programming, functions are used for both. (even more confusing than in Prolog if you think about it)

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- **Example 2.19.** Consider again the reverse predicate for lists below:
  An input datum is e.g. [1,2,3], then the output datum is [3,2,1].

```
reverse([],[]).
reverse([X|R],L):−reverse(R,S),append(S,[X],L).
```

We "define" the computational behavior of the predicate rev, but the list constructors [...] are just used to construct lists from arguments.

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  We "define" the computational behavior of the predicate rev, but the list constructors [. . .] are just used to construct lists from arguments.
- **Example 2.24 (Trees and Leaf Counting).** We represent (unlabelled) trees via the function t from tree lists to trees. For instance, a balanced binary tree of depth 2 is t([t([t([]),t([])]),t([t([]),t([])])]). We count leaves by

```
leafcount(t([]),1).
leafcount(t([V]),W) :− leafcount(V,W).
leafcount(t([X|R]),Y) :− leafcount(X,Z), leafcount(t(R),W), Y is Z + W.
```

For more information on Prolog

RTFM ($\widehat{=}$ "read the fine manuals")

- **RTFM Resources:** There are also lots of good tutorials on the web,
  - I personally like [Fis; LPN],
  - [Fla94] has a very thorough logic-based introduction,
  - consult also the SWI Prolog Manual [SWI],

Chapter 4
Recap of Prerequisites from Math & Theoretical Computer Science

4.1 Recap: Complexity Analysis in AI?

Performance and Scaling

- Suppose we have three algorithms to choose from. (which one to select)
- Systematic analysis reveals performance characteristics.
- **Example 1.1.** For a computational problem of size n we have

	performance		
size	linear	quadratic	exponential
n	$100n\mu\mathrm{s}$	$7n^2\mu\mathrm{s}$	$2^n\mu\mathrm{s}$
1	$100\mu\mathrm{s}$	$7\mu\mathrm{s}$	$2\mu\mathrm{s}$
5	$.5\mathrm{ms}$	$175\mu\mathrm{s}$	$32\mu\mathrm{s}$
10	$1\mathrm{ms}$	$.7\mathrm{ms}$	$1\mathrm{ms}$
45	$4.5\mathrm{ms}$	$14\mathrm{ms}$	$1.1Y$
100	...	...	...
1 000	...	...	...
10 000	...	...	...
1 000 000	...	...	...

What?! One year?

- $2^{10} = 1\,024$ $(1024\mu\text{s} \simeq 1\text{ms})$
- $2^{45} = 35\,184\,372\,088\,832$ $(3.5{\times}10^{13}\mu\text{s} \simeq 3.5{\times}10^{7}\text{s} \simeq 1.1Y)$
- **Example 1.2.** We denote all times that are longer than the age of the universe with —

	performance		
size	linear	quadratic	exponential
n	$100n\mu\text{s}$	$7n^2\mu\text{s}$	$2^n\mu\text{s}$
1	$100\mu\text{s}$	$7\mu\text{s}$	$2\mu\text{s}$
5	$.5\text{ms}$	$175\mu\text{s}$	$32\mu\text{s}$
10	1ms	$.7\text{ms}$	1ms
45	4.5ms	14ms	$1.1Y$
< 100	100ms	7s	$10^{16}Y$
$1\,000$	1s	12min	—
$10\,000$	10s	20h	—
$1\,000\,000$	1.6min	2.5mon	—

Recap: Time/Space Complexity of Algorithms

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- **Definition 1.6.** We say that an algorithm α that terminates in time $t(n)$ for all inputs of size n has running time $T(\alpha) := t$.
  Let $S \subseteq \mathbb{N} \to \mathbb{N}$ be a set of natural number functions, then we say that α has time complexity in S (written $T(\alpha) \in S$ or colloquially $T(\alpha) = S$), iff $t \in S$. We say α has space complexity in S, iff α uses only memory of size $s(n)$ on inputs of size n and $s \in S$.

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- **Definition 1.13.** The following sets are often used for S in $T(\alpha)$:

Landau set	class name	rank	Landau set	class name	rank
$\mathcal{O}(1)$	constant	1	$\mathcal{O}(n^2)$	quadratic	4
$\mathcal{O}(\log_2(n))$	logarithmic	2	$\mathcal{O}(n^k)$	polynomial	5
$\mathcal{O}(n)$	linear	3	$\mathcal{O}(k^n)$	exponential	6

  where $\mathcal{O}(g) = \{f \mid \exists k > 0. f \leq_a k \cdot g\}$ and $f \leq_a g$ (f is asymptotically bounded by g), iff there is an $n_0 \in \mathbb{N}$, such that $f(n) \leq g(n)$ for all $n > n_0$.

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- **Lemma 1.17 (Growth Ranking).** *For $k' > 2$ and $k > 1$ we have*

$$\mathcal{O}(1) \subset \mathcal{O}(\log_2(n)) \subset \mathcal{O}(n) \subset \mathcal{O}(n^2) \subset \mathcal{O}(n^{k'}) \subset \mathcal{O}(k^n)$$

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$$\mathcal{O}(1) \subset \mathcal{O}(\log_2(n)) \subset \mathcal{O}(n) \subset \mathcal{O}(n^2) \subset \mathcal{O}(n^{k'}) \subset \mathcal{O}(k^n)$$

- **For AI-1:** I expect that given an algorithm, you can determine its complexity class. (next)

Advantage: Big-Oh Arithmetics

- **Practical Advantage:** Computing with Landau sets is quite simple. (good simplification)
- **Theorem 1.21 (Computing with Landau Sets).**
  1. *If $\mathcal{O}(c \cdot f) = \mathcal{O}(f)$ for any constant $c \in \mathbb{N}$.* *(drop constant factors)*
  2. *If $\mathcal{O}(f) \subseteq \mathcal{O}(g)$, then $\mathcal{O}(f + g) = \mathcal{O}(g)$.* *(drop low-complexity summands)*
  3. *If $\mathcal{O}(f \cdot g) = \mathcal{O}(f) \cdot \mathcal{O}(g)$.* *(distribute over products)*
- These are not all of "big-Oh calculation rules", but they're enough for most purposes
- **Applications:** Convince yourselves using the result above that
  - $\mathcal{O}(4n^3 + 3n + 7^{1000n}) = \mathcal{O}(2^n)$
  - $\mathcal{O}(n) \subset \mathcal{O}(n \cdot \log_2(n)) \subset \mathcal{O}(n^2)$

Determining the Time/Space Complexity of Algorithms

- **Definition 1.22.** Given a function Γ that assigns variables v to functions $\Gamma(v)$ and α an imperative algorithm, we compute the
  - time complexity $T_\Gamma(\alpha)$ of program α and
  - the context $C_\Gamma(\alpha)$ introduced by α

  by joint induction on the structure of α:
  - constant: can be accessed in constant time

Determining the Time/Space Complexity of Algorithms

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  - time complexity $T_\Gamma(\alpha)$ of program α and
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  by joint induction on the structure of α:
  - constant: If $\alpha = \delta$ for a data constant δ, then $T_\Gamma(\alpha) \in \mathcal{O}(1)$.

Determining the Time/Space Complexity of Algorithms

- **Definition 1.24.** Given a function Γ that assigns variables v to functions $\Gamma(v)$ and α an imperative algorithm, we compute the
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  by joint induction on the structure of α:
  - constant: If $\alpha = \delta$ for a data constant δ, then $T_\Gamma(\alpha) \in \mathcal{O}(1)$.
  - variable: need the complexity of the value

Determining the Time/Space Complexity of Algorithms

- **Definition 1.25.** Given a function Γ that assigns variables v to functions $\Gamma(v)$ and α an imperative algorithm, we compute the
  - time complexity $T_\Gamma(\alpha)$ of program α and
  - the context $C_\Gamma(\alpha)$ introduced by α

  by joint induction on the structure of α:
  - constant: If $\alpha = \delta$ for a data constant δ, then $T_\Gamma(\alpha) \in \mathcal{O}(1)$.
  - variable: If $\alpha = v$ with $v \in \mathrm{dom}(\Gamma)$, then $T_\Gamma(\alpha) \in \mathcal{O}(\Gamma(v))$.

Determining the Time/Space Complexity of Algorithms

- **Definition 1.26.** Given a function Γ that assigns variables v to functions $\Gamma(v)$ and α an imperative algorithm, we compute the
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  by joint induction on the structure of α:
  - constant: If $\alpha = \delta$ for a data constant δ, then $T_\Gamma(\alpha) \in \mathcal{O}(1)$.
  - variable: If $\alpha = v$ with $v \in \mathrm{dom}(\Gamma)$, then $T_\Gamma(\alpha) \in \mathcal{O}(\Gamma(v))$.
  - application: compose the complexities of the function and the argument

Determining the Time/Space Complexity of Algorithms

- **Definition 1.27.** Given a function Γ that assigns variables v to functions $\Gamma(v)$ and α an imperative algorithm, we compute the
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  by joint induction on the structure of α:
  - constant: If $\alpha = \delta$ for a data constant δ, then $T_\Gamma(\alpha) \in \mathcal{O}(1)$.
  - variable: If $\alpha = v$ with $v \in \mathrm{dom}(\Gamma)$, then $T_\Gamma(\alpha) \in \mathcal{O}(\Gamma(v))$.
  - application: If $\alpha = \varphi(\psi)$ with $T_\Gamma(\varphi) \in \mathcal{O}(f)$ and $T_{\Gamma \cup C_\Gamma(\varphi)}(\psi) \in \mathcal{O}(g)$, then $T_\Gamma(\alpha) \in \mathcal{O}(f \circ g)$ and $C_\Gamma(\alpha) = C_{\Gamma \cup C_\Gamma(\varphi)}(\psi)$.

Determining the Time/Space Complexity of Algorithms

- **Definition 1.28.** Given a function Γ that assigns variables v to functions $\Gamma(v)$ and α an imperative algorithm, we compute the
  - time complexity $T_\Gamma(\alpha)$ of program α and
  - the context $C_\Gamma(\alpha)$ introduced by α

  by joint induction on the structure of α:
  - constant: If $\alpha = \delta$ for a data constant δ, then $T_\Gamma(\alpha) \in \mathcal{O}(1)$.
  - variable: If $\alpha = v$ with $v \in \mathrm{dom}(\Gamma)$, then $T_\Gamma(\alpha) \in \mathcal{O}(\Gamma(v))$.
  - application: If $\alpha = \varphi(\psi)$ with $T_\Gamma(\varphi) \in \mathcal{O}(f)$ and $T_{\Gamma \cup C_\Gamma(\varphi)}(\psi) \in \mathcal{O}(g)$, then $T_\Gamma(\alpha) \in \mathcal{O}(f \circ g)$ and $C_\Gamma(\alpha) = C_{\Gamma \cup C_\Gamma(\varphi)}(\psi)$.
  - assignment: has to compute the value $\rightsquigarrow$ has its complexity

Determining the Time/Space Complexity of Algorithms

- **Definition 1.29.** Given a function Γ that assigns variables v to functions $\Gamma(v)$ and α an imperative algorithm, we compute the
  - time complexity $T_\Gamma(\alpha)$ of program α and
  - the context $C_\Gamma(\alpha)$ introduced by α

  by joint induction on the structure of α:
  - constant: If $\alpha = \delta$ for a data constant δ, then $T_\Gamma(\alpha) \in \mathcal{O}(1)$.
  - variable: If $\alpha = v$ with $v \in \mathrm{dom}(\Gamma)$, then $T_\Gamma(\alpha) \in \mathcal{O}(\Gamma(v))$.
  - application: If $\alpha = \varphi(\psi)$ with $T_\Gamma(\varphi) \in \mathcal{O}(f)$ and $T_{\Gamma \cup C_\Gamma(\varphi)}(\psi) \in \mathcal{O}(g)$, then $T_\Gamma(\alpha) \in \mathcal{O}(f \circ g)$ and $C_\Gamma(\alpha) = C_{\Gamma \cup C_\Gamma(\varphi)}(\psi)$.
  - assignment: If α is $v := \varphi$ with $T_\Gamma(\varphi) \in S$, then $T_\Gamma(\alpha) \in S$ and $C_\Gamma(\alpha) = \Gamma \cup (v, S)$.

Determining the Time/Space Complexity of Algorithms

- **Definition 1.30.** Given a function Γ that assigns variables v to functions $\Gamma(v)$ and α an imperative algorithm, we compute the
  - time complexity $T_\Gamma(\alpha)$ of program α and
  - the context $C_\Gamma(\alpha)$ introduced by α

  by joint induction on the structure of α:
  - constant: If $\alpha = \delta$ for a data constant δ, then $T_\Gamma(\alpha) \in \mathcal{O}(1)$.
  - variable: If $\alpha = v$ with $v \in \mathrm{dom}(\Gamma)$, then $T_\Gamma(\alpha) \in \mathcal{O}(\Gamma(v))$.
  - application: If $\alpha = \varphi(\psi)$ with $T_\Gamma(\varphi) \in \mathcal{O}(f)$ and $T_{\Gamma \cup C_\Gamma(\varphi)}(\psi) \in \mathcal{O}(g)$, then $T_\Gamma(\alpha) \in \mathcal{O}(f \circ g)$ and $C_\Gamma(\alpha) = C_{\Gamma \cup C_\Gamma(\varphi)}(\psi)$.
  - assignment: If α is $v{:=}\varphi$ with $T_\Gamma(\varphi) \in S$, then $T_\Gamma(\alpha) \in S$ and $C_\Gamma(\alpha) = \Gamma \cup (v, S)$.
  - composition: has the maximal complexity of the components

Determining the Time/Space Complexity of Algorithms

- **Definition 1.31.** Given a function Γ that assigns variables v to functions $\Gamma(v)$ and α an imperative algorithm, we compute the
  - time complexity $T_\Gamma(\alpha)$ of program α and
  - the context $C_\Gamma(\alpha)$ introduced by α

  by joint induction on the structure of α:
  - constant: If $\alpha = \delta$ for a data constant δ, then $T_\Gamma(\alpha) \in \mathcal{O}(1)$.
  - variable: If $\alpha = v$ with $v \in \mathrm{dom}(\Gamma)$, then $T_\Gamma(\alpha) \in \mathcal{O}(\Gamma(v))$.
  - application: If $\alpha = \varphi(\psi)$ with $T_\Gamma(\varphi) \in \mathcal{O}(f)$ and $T_{\Gamma \cup C_\Gamma(\varphi)}(\psi) \in \mathcal{O}(g)$, then $T_\Gamma(\alpha) \in \mathcal{O}(f \circ g)$ and $C_\Gamma(\alpha) = C_{\Gamma \cup C_\Gamma(\varphi)}(\psi)$.
  - assignment: If α is $v := \varphi$ with $T_\Gamma(\varphi) \in S$, then $T_\Gamma(\alpha) \in S$ and $C_\Gamma(\alpha) = \Gamma \cup (v,S)$.
  - composition: If α is $\varphi\,;\,\psi$, with $T_\Gamma(\varphi) \in P$ and $T_{\Gamma \cup C_\Gamma(\psi)}(\psi) \in Q$, then $T_\Gamma(\alpha) \in \max\{P, Q\}$ and $C_\Gamma(\alpha) = C_{\Gamma \cup C_\Gamma(\psi)}(\psi)$.

Determining the Time/Space Complexity of Algorithms

- **Definition 1.32.** Given a function Γ that assigns variables v to functions $\Gamma(v)$ and α an imperative algorithm, we compute the
  - time complexity $T_\Gamma(\alpha)$ of program α and
  - the context $C_\Gamma(\alpha)$ introduced by α

  by joint induction on the structure of α:
  - constant: If $\alpha = \delta$ for a data constant δ, then $T_\Gamma(\alpha) \in \mathcal{O}(1)$.
  - variable: If $\alpha = v$ with $v \in \mathrm{dom}(\Gamma)$, then $T_\Gamma(\alpha) \in \mathcal{O}(\Gamma(v))$.
  - application: If $\alpha = \varphi(\psi)$ with $T_\Gamma(\varphi) \in \mathcal{O}(f)$ and $T_{\Gamma \cup C_\Gamma(\varphi)}(\psi) \in \mathcal{O}(g)$, then $T_\Gamma(\alpha) \in \mathcal{O}(f \circ g)$ and $C_\Gamma(\alpha) = C_{\Gamma \cup C_\Gamma(\varphi)}(\psi)$.
  - assignment: If α is $v := \varphi$ with $T_\Gamma(\varphi) \in S$, then $T_\Gamma(\alpha) \in S$ and $C_\Gamma(\alpha) = \Gamma \cup (v, S)$.
  - composition: If α is $\varphi \,;\, \psi$, with $T_\Gamma(\varphi) \in P$ and $T_{\Gamma \cup C_\Gamma(\psi)}(\psi) \in Q$, then $T_\Gamma(\alpha) \in \max\{P, Q\}$ and $C_\Gamma(\alpha) = C_{\Gamma \cup C_\Gamma(\psi)}(\psi)$.
  - branching: has the maximal complexity of the condition and branches

Determining the Time/Space Complexity of Algorithms

- **Definition 1.33.** Given a function Γ that assigns variables v to functions $\Gamma(v)$ and α an imperative algorithm, we compute the
  - time complexity $T_\Gamma(\alpha)$ of program α and
  - the context $C_\Gamma(\alpha)$ introduced by α

  by joint induction on the structure of α:
  - constant: If $\alpha = \delta$ for a data constant δ, then $T_\Gamma(\alpha) \in \mathcal{O}(1)$.
  - variable: If $\alpha = v$ with $v \in \mathrm{dom}(\Gamma)$, then $T_\Gamma(\alpha) \in \mathcal{O}(\Gamma(v))$.
  - application: If $\alpha = \varphi(\psi)$ with $T_\Gamma(\varphi) \in \mathcal{O}(f)$ and $T_{\Gamma \cup C_\Gamma(\varphi)}(\psi) \in \mathcal{O}(g)$, then $T_\Gamma(\alpha) \in \mathcal{O}(f \circ g)$ and $C_\Gamma(\alpha) = C_{\Gamma \cup C_\Gamma(\varphi)}(\psi)$.
  - assignment: If α is $v := \varphi$ with $T_\Gamma(\varphi) \in S$, then $T_\Gamma(\alpha) \in S$ and $C_\Gamma(\alpha) = \Gamma \cup (v, S)$.
  - composition: If α is $\varphi\,;\psi$, with $T_\Gamma(\varphi) \in P$ and $T_{\Gamma \cup C_\Gamma(\psi)}(\psi) \in Q$, then $T_\Gamma(\alpha) \in \max\{P, Q\}$ and $C_\Gamma(\alpha) = C_{\Gamma \cup C_\Gamma(\psi)}(\psi)$.
  - branching: If α is $\mathbf{if}\gamma\mathbf{then}\varphi\mathbf{else}\psi\mathbf{end}$, with $T_\Gamma(\gamma) \in C$, $T_{\Gamma \cup C_\Gamma(\gamma)}(\varphi) \in P$, $T_{\Gamma \cup C_\Gamma(\gamma)}(\varphi) \in Q$, and then $T_\Gamma(\alpha) \in \max\{C, P, Q\}$ and $C_\Gamma(\alpha) = \Gamma \cup C_\Gamma(\gamma) \cup C_{\Gamma \cup C_\Gamma(\gamma)}(\varphi) \cup C_{\Gamma \cup C_\Gamma(\gamma)}(\psi)$.

Determining the Time/Space Complexity of Algorithms

- **Definition 1.34.** Given a function Γ that assigns variables v to functions $\Gamma(v)$ and α an imperative algorithm, we compute the
  - time complexity $T_\Gamma(\alpha)$ of program α and
  - the context $C_\Gamma(\alpha)$ introduced by α

  by joint induction on the structure of α:
  - constant: If $\alpha = \delta$ for a data constant δ, then $T_\Gamma(\alpha) \in \mathcal{O}(1)$.
  - variable: If $\alpha = v$ with $v \in \mathrm{dom}(\Gamma)$, then $T_\Gamma(\alpha) \in \mathcal{O}(\Gamma(v))$.
  - application: If $\alpha = \varphi(\psi)$ with $T_\Gamma(\varphi) \in \mathcal{O}(f)$ and $T_{\Gamma \cup C_\Gamma(\varphi)}(\psi) \in \mathcal{O}(g)$, then $T_\Gamma(\alpha) \in \mathcal{O}(f \circ g)$ and $C_\Gamma(\alpha) = C_{\Gamma \cup C_\Gamma(\varphi)}(\psi)$.
  - assignment: If α is $v{:=}\varphi$ with $T_\Gamma(\varphi) \in S$, then $T_\Gamma(\alpha) \in S$ and $C_\Gamma(\alpha) = \Gamma \cup (v, S)$.
  - composition: If α is $\varphi\,;\psi$, with $T_\Gamma(\varphi) \in P$ and $T_{\Gamma \cup C_\Gamma(\psi)}(\psi) \in Q$, then $T_\Gamma(\alpha) \in \max\{P, Q\}$ and $C_\Gamma(\alpha) = C_{\Gamma \cup C_\Gamma(\psi)}(\psi)$.
  - branching: If α is **if**γ**then**φ**else**ψ**end**, with $T_\Gamma(\gamma) \in C$, $T_{\Gamma \cup C_\Gamma(\gamma)}(\varphi) \in P$, $T_{\Gamma \cup C_\Gamma(\gamma)}(\varphi) \in Q$, and then $T_\Gamma(\alpha) \in \max\{C, P, Q\}$ and $C_\Gamma(\alpha) = \Gamma \cup C_\Gamma(\gamma) \cup C_{\Gamma \cup C_\Gamma(\gamma)}(\varphi) \cup C_{\Gamma \cup C_\Gamma(\gamma)}(\psi)$.
  - looping: multiplies complexities

Determining the Time/Space Complexity of Algorithms

- **Definition 1.35.** Given a function Γ that assigns variables v to functions $\Gamma(v)$ and α an imperative algorithm, we compute the
  - time complexity $T_\Gamma(\alpha)$ of program α and
  - the context $C_\Gamma(\alpha)$ introduced by α

  by joint induction on the structure of α:
  - constant: If $\alpha = \delta$ for a data constant δ, then $T_\Gamma(\alpha) \in \mathcal{O}(1)$.
  - variable: If $\alpha = v$ with $v \in \mathrm{dom}(\Gamma)$, then $T_\Gamma(\alpha) \in \mathcal{O}(\Gamma(v))$.
  - application: If $\alpha = \varphi(\psi)$ with $T_\Gamma(\varphi) \in \mathcal{O}(f)$ and $T_{\Gamma \cup C_\Gamma(\varphi)}(\psi) \in \mathcal{O}(g)$, then $T_\Gamma(\alpha) \in \mathcal{O}(f \circ g)$ and $C_\Gamma(\alpha) = C_{\Gamma \cup C_\Gamma(\varphi)}(\psi)$.
  - assignment: If α is $v := \varphi$ with $T_\Gamma(\varphi) \in S$, then $T_\Gamma(\alpha) \in S$ and $C_\Gamma(\alpha) = \Gamma \cup (v, S)$.
  - composition: If α is $\varphi\,;\,\psi$, with $T_\Gamma(\varphi) \in P$ and $T_{\Gamma \cup C_\Gamma(\psi)}(\psi) \in Q$, then $T_\Gamma(\alpha) \in \max\{P, Q\}$ and $C_\Gamma(\alpha) = C_{\Gamma \cup C_\Gamma(\psi)}(\psi)$.
  - branching: If α is $\mathbf{if}\,\gamma\,\mathbf{then}\,\varphi\,\mathbf{else}\,\psi\,\mathbf{end}$, with $T_\Gamma(\gamma) \in C$, $T_{\Gamma \cup C_\Gamma(\gamma)}(\varphi) \in P$, $T_{\Gamma \cup C_\Gamma(\gamma)}(\varphi) \in Q$, and then $T_\Gamma(\alpha) \in \max\{C, P, Q\}$ and $C_\Gamma(\alpha) = \Gamma \cup C_\Gamma(\gamma) \cup C_{\Gamma \cup C_\Gamma(\gamma)}(\varphi) \cup C_{\Gamma \cup C_\Gamma(\gamma)}(\psi)$.
  - looping: If α is $\mathbf{while}\,\gamma\,\mathbf{do}\,\varphi\,\mathbf{end}$, with $T_\Gamma(\gamma) \in \mathcal{O}(f)$, $T_{\Gamma \cup C_\Gamma(\gamma)}(\varphi) \in \mathcal{O}(g)$, then $T_\Gamma(\alpha) \in \mathcal{O}(f(n) \cdot g(n))$ and $C_\Gamma(\alpha) = C_{\Gamma \cup C_\Gamma(\gamma)}(\varphi)$.

Determining the Time/Space Complexity of Algorithms

- **Definition 1.36.** Given a function Γ that assigns variables v to functions $\Gamma(v)$ and α an imperative algorithm, we compute the
  - time complexity $T_\Gamma(\alpha)$ of program α and
  - the context $C_\Gamma(\alpha)$ introduced by α

  by joint induction on the structure of α:
  - constant: If $\alpha = \delta$ for a data constant δ, then $T_\Gamma(\alpha) \in \mathcal{O}(1)$.
  - variable: If $\alpha = v$ with $v \in \mathrm{dom}(\Gamma)$, then $T_\Gamma(\alpha) \in \mathcal{O}(\Gamma(v))$.
  - application: If $\alpha = \varphi(\psi)$ with $T_\Gamma(\varphi) \in \mathcal{O}(f)$ and $T_{\Gamma \cup C_\Gamma(\varphi)}(\psi) \in \mathcal{O}(g)$, then $T_\Gamma(\alpha) \in \mathcal{O}(f \circ g)$ and $C_\Gamma(\alpha) = C_{\Gamma \cup C_\Gamma(\varphi)}(\psi)$.
  - assignment: If α is $v := \varphi$ with $T_\Gamma(\varphi) \in S$, then $T_\Gamma(\alpha) \in S$ and $C_\Gamma(\alpha) = \Gamma \cup (v, S)$.
  - composition: If α is $\varphi\,;\psi$, with $T_\Gamma(\varphi) \in P$ and $T_{\Gamma \cup C_\Gamma(\psi)}(\psi) \in Q$, then $T_\Gamma(\alpha) \in \max\{P, Q\}$ and $C_\Gamma(\alpha) = C_{\Gamma \cup C_\Gamma(\psi)}(\psi)$.
  - branching: If α is $\mathbf{if}\,\gamma\,\mathbf{then}\,\varphi\,\mathbf{else}\,\psi\,\mathbf{end}$, with $T_\Gamma(\gamma) \in C$, $T_{\Gamma \cup C_\Gamma(\gamma)}(\varphi) \in P$, $T_{\Gamma \cup C_\Gamma(\gamma)}(\varphi) \in Q$, and then $T_\Gamma(\alpha) \in \max\{C, P, Q\}$ and $C_\Gamma(\alpha) = \Gamma \cup C_\Gamma(\gamma) \cup C_{\Gamma \cup C_\Gamma(\gamma)}(\varphi) \cup C_{\Gamma \cup C_\Gamma(\gamma)}(\psi)$.
  - looping: If α is $\mathbf{while}\,\gamma\,\mathbf{do}\,\varphi\,\mathbf{end}$, with $T_\Gamma(\gamma) \in \mathcal{O}(f)$, $T_{\Gamma \cup C_\Gamma(\gamma)}(\varphi) \in \mathcal{O}(g)$, then $T_\Gamma(\alpha) \in \mathcal{O}(f(n) \cdot g(n))$ and $C_\Gamma(\alpha) = C_{\Gamma \cup C_\Gamma(\gamma)}(\varphi)$.
  - The time complexity $T(\alpha)$ is just $T_\emptyset(\alpha)$, where $\emptyset$ is the empty function.

Determining the Time/Space Complexity of Algorithms

- **Definition 1.37.** Given a function Γ that assigns variables v to functions $\Gamma(v)$ and α an imperative algorithm, we compute the
  - time complexity $T_\Gamma(\alpha)$ of program α and
  - the context $C_\Gamma(\alpha)$ introduced by α

  by joint induction on the structure of α:
  - constant: If $\alpha = \delta$ for a data constant δ, then $T_\Gamma(\alpha) \in \mathcal{O}(1)$.
  - variable: If $\alpha = v$ with $v \in \mathrm{dom}(\Gamma)$, then $T_\Gamma(\alpha) \in \mathcal{O}(\Gamma(v))$.
  - application: If $\alpha = \varphi(\psi)$ with $T_\Gamma(\varphi) \in \mathcal{O}(f)$ and $T_{\Gamma \cup C_\Gamma(\varphi)}(\psi) \in \mathcal{O}(g)$, then $T_\Gamma(\alpha) \in \mathcal{O}(f \circ g)$ and $C_\Gamma(\alpha) = C_{\Gamma \cup C_\Gamma(\varphi)}(\psi)$.
  - assignment: If α is $v := \varphi$ with $T_\Gamma(\varphi) \in S$, then $T_\Gamma(\alpha) \in S$ and $C_\Gamma(\alpha) = \Gamma \cup (v,S)$.
  - composition: If α is $\varphi\,;\psi$, with $T_\Gamma(\varphi) \in P$ and $T_{\Gamma \cup C_\Gamma(\psi)}(\psi) \in Q$, then $T_\Gamma(\alpha) \in \max\,\{P, Q\}$ and $C_\Gamma(\alpha) = C_{\Gamma \cup C_\Gamma(\psi)}(\psi)$.
  - branching: If α is $\mathbf{if}\gamma\mathbf{then}\varphi\mathbf{else}\psi\mathbf{end}$, with $T_\Gamma(\gamma) \in C$, $T_{\Gamma \cup C_\Gamma(\gamma)}(\varphi) \in P$, $T_{\Gamma \cup C_\Gamma(\gamma)}(\varphi) \in Q$, and then $T_\Gamma(\alpha) \in \max\,\{C, P, Q\}$ and $C_\Gamma(\alpha) = \Gamma \cup C_\Gamma(\gamma) \cup C_{\Gamma \cup C_\Gamma(\gamma)}(\varphi) \cup C_{\Gamma \cup C_\Gamma(\gamma)}(\psi)$.
  - looping: If α is $\mathbf{while}\gamma\mathbf{do}\varphi\mathbf{end}$, with $T_\Gamma(\gamma) \in \mathcal{O}(f)$, $T_{\Gamma \cup C_\Gamma(\gamma)}(\varphi) \in \mathcal{O}(g)$, then $T_\Gamma(\alpha) \in \mathcal{O}(f(n) \cdot g(n))$ and $C_\Gamma(\alpha) = C_{\Gamma \cup C_\Gamma(\gamma)}(\varphi)$.
  - The time complexity $T(\alpha)$ is just $T_\emptyset(\alpha)$, where $\emptyset$ is the empty function.
- Recursion is much more difficult to analyze $\leadsto$ recurrences and Master's theorem.

Why Complexity Analysis? (General)

- **Example 1.38.** Once upon a time I was trying to invent an efficient algorithm.
  - My first algorithm attempt didn't work, so I had to try harder.

Why Complexity Analysis? (General)

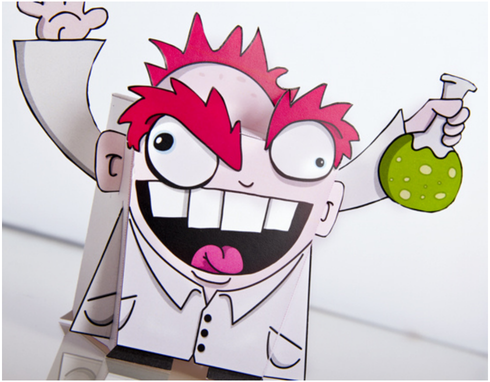
- **Example 1.39.** Once upon a time I was trying to invent an efficient algorithm.
  - My first algorithm attempt didn't work, so I had to try harder.
  - But my 2nd attempt didn't work either, which got me a bit agitated.

Why Complexity Analysis? (General)

- **Example 1.40.** Once upon a time I was trying to invent an efficient algorithm.
  - My first algorithm attempt didn't work, so I had to try harder.
  - But my 2nd attempt didn't work either, which got me a bit agitated.
  - The 3rd attempt didn't work either...

Why Complexity Analysis? (General)

- **Example 1.41.** Once upon a time I was trying to invent an efficient algorithm.
  - My first algorithm attempt didn't work, so I had to try harder.
  - But my 2nd attempt didn't work either, which got me a bit agitated.
  - The 3rd attempt didn't work either. . .
  - And neither the 4th. But then:



Why Complexity Analysis? (General)

- **Example 1.42.** Once upon a time I was trying to invent an efficient algorithm.
  - My first algorithm attempt didn't work, so I had to try harder.
  - But my 2nd attempt didn't work either, which got me a bit agitated.
  - The 3rd attempt didn't work either...
  - And neither the 4th. But then:
  - Ta-da ...when, for once, I turned around and looked in the other direction– CAN one actually solve this efficiently? – NP hardness was there to rescue me.

Why Complexity Analysis? (General)

- **Example 1.43.** Trying to find a sea route east to India (from Spain) (does not exist)

- **Observation:** Complexity theory saves you from spending lots of time trying to invent algorithms that do not exist.

Reminder (?): NP and PSPACE (details $\rightsquigarrow$ e.g. [GJ79])

- **Turing Machine:** Works on a tape consisting of cells, across which its Read/Write head moves. The machine has internal states. There is a transition function that specifies – given the current cell content and internal state – what the subsequent internal state will be, how what the R/W head does (write a symbol and/or move). Some internal states are accepting.
- Decision problems are in NP if there is a non deterministic Turing machine that halts with an answer after time polynomial in the size of its input. Accepts if *at least one* of the possible runs accepts.
- Decision problems are in NPSPACE, if there is a non deterministic Turing machine that runs in space polynomial in the size of its input.
- **NP vs. PSPACE:** Non-deterministic polynomial space can be simulated in deterministic polynomial space. Thus PSPACE = NPSPACE, and hence (trivially) NP $\subseteq$ PSPACE.
  It is commonly believed that NP$\not\supseteq$PSPACE. (similar to P $\subseteq$ NP)

The Utility of Complexity Knowledge (NP-Hardness)

- **Assume:** In 3 years from now, you have finished your studies and are working in your first industry job. Your boss Mr. X gives you a problem and says *Solve It!*. By which he means, *write a program that solves it efficiently*.
- **Question:** Assume further that, after trying in vain for 4 weeks, you got the next meeting with Mr. X. How could knowing about NP hardness help?

The Utility of Complexity Knowledge (NP-Hardness)

- **Assume:** In 3 years from now, you have finished your studies and are working in your first industry job. Your boss Mr. X gives you a problem and says *Solve It!*. By which he means, *write a program that solves it efficiently*.
- **Question:** Assume further that, after trying in vain for 4 weeks, you got the next meeting with Mr. X. How could knowing about NP hardness help?
- **Answer:** It helps you save your skin with (theoretical computer) science!
  - Do you want to say *Um, sorry, but I couldn't find an efficient solution, please don't fire me*?
  - Or would you rather say *Look, I didn't find an efficient solution. But neither could all the Turing-award winners out there put together, because the problem is NP hard*?

4.2 Recap: Formal Languages and Grammars

The Mathematics of Strings

- **Definition 2.1.** An alphabet A is a finite set; we call each element $a \in A$ a character, and an n tuple $s \in A^n$ a string (of length n over A).
- **Definition 2.2.** Note that $A^0 = \{\langle\rangle\}$, where $\langle\rangle$ is the (unique) 0-tuple. With the definition above we consider $\langle\rangle$ as the string of length 0 and call it the empty string and denote it with ϵ.
- **Note:** Sets $\neq$ strings, e.g. $\{1, 2, 3\} = \{3, 2, 1\}$, but $\langle 1, 2, 3\rangle \neq \langle 3, 2, 1\rangle$.
- **Notation:** We will often write a string $\langle c_1, \ldots, c_n\rangle$ as $"\mathtt{c_1 \ldots c_n}"$, for instance $"\mathtt{abc}"$ for $\langle \mathtt{a}, \mathtt{b}, \mathtt{c}\rangle$
- **Example 2.3.** Take $A = \{\mathtt{h}, \mathtt{1}, /\}$ as an alphabet. Each of the members $\mathtt{h}$, $\mathtt{1}$, and $/$ is a character. The vector $\langle /, /, \mathtt{1}, \mathtt{h}, \mathtt{1}\rangle$ is a string of length 5 over A.
- **Definition 2.4 (String Length).** Given a string s we denote its length with $|s|$.
- **Definition 2.5.** The concatenation $\mathrm{conc}(s, t)$ of two strings $s = \langle s_1, \ldots, s_n\rangle \in A^n$ and $t = \langle t_1, \ldots, t_m\rangle \in A^m$ is defined as $\langle s_1, \ldots, s_n, t_1, \ldots, t_m\rangle \in A^{n+m}$.
  We will often write $\mathrm{conc}(s, t)$ as $s + t$ or simply st
- **Example 2.6.** $\mathrm{conc}("\mathtt{text}", "\mathtt{book}") = "\mathtt{text}" + "\mathtt{book}" = "\mathtt{textbook}"$

Formal Languages

- **Definition 2.7.** Let A be an alphabet, then we define the sets $A^+ := \bigcup_{i \in \mathbb{N}^+} A^i$ of nonempty string and $A^* := A^+ \cup \{\epsilon\}$ of strings.
- **Example 2.8.** If $A = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$, then $A^* = \{\epsilon, \mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{aa}, \mathtt{ab}, \mathtt{ac}, \mathtt{ba}, \ldots, \mathtt{aaa}, \ldots\}$.
- **Definition 2.9.** A set $L \subseteq A^*$ is called a formal language over A.
- **Definition 2.10.** We use $\mathtt{c}^{[n]}$ for the string that consists of the character $\mathtt{c}$ repeated n times.
- **Example 2.11.** $\#^{[5]} = \langle \#, \#, \#, \#, \# \rangle$
- **Example 2.12.** The set $M := \{\mathtt{b}\mathtt{a}^{[n]} \mid n \in \mathbb{N}\}$ of strings that start with character $\mathtt{b}$ followed by an arbitrary numbers of $\mathtt{a}$'s is a formal language over $A = \{\mathtt{a}, \mathtt{b}\}$.
- **Definition 2.13.** Let $L_1, L_2, L \subseteq \Sigma^*$ be formal languages over Σ.
  - Intersection and union: $L_1 \cap L_2$, $L_1 \cup L_2$.
  - Language complement L: $\overline{L} := \Sigma^* \backslash L$.
  - The language concatenation of L_1 and L_2: $L_1 L_2 := \{uw \mid u \in L_1,\ w \in L_2\}$. We often use $L_1 L_2$ instead of $L_1 L_2$.
  - Language power L: $L^0 := \{\epsilon\}$, $L^{n+1} := L L^n$, where $L^n := \{\mathtt{w}_1 \ldots \mathtt{w}_n \mid w_i \in L$, for $i = 1 \ldots n\}$, (for $n \in \mathbb{N}$).
  - language Kleene closure L: $L^* := \bigcup_{n \in \mathbb{N}} L^n$ and also $L^+ := \bigcup_{n \in \mathbb{N}^+} L^n$.
  - The reflection of a language L: $L^R := \{w^R \mid w \in L\}$.

Phrase Structure Grammars (Theory)

- **Recap:** A formal language is an arbitrary set of symbol sequences.
- **Problem:** This may be infinite and even undecidable even if A is finite.
- **Idea:** Find a way of representing formal languages with structure finitely.
- **Definition 2.14.** A phrase structure grammar (also called type 0 grammar, unrestricted grammar, or just grammar) is a tuple $\langle N, \Sigma, P, S\rangle$ where
  - N is a finite set of nonterminal symbols,
  - Σ is a finite set of terminal symbols, members of $\Sigma \cup N$ are called symbols.
  - P is a finite set of production rules: pairs $p := h \rightarrow b$ (also written as $h \Rightarrow b$), where $h \in (\Sigma \cup N)^* N (\Sigma \cup N)^*$ and $b \in (\Sigma \cup N)^*$. The string h is called the head of p and b the body.
  - $S \in N$ is a distinguished symbol called the start symbol (also sentence symbol).

  The sets N and Σ are assumed to be disjoint. Any word $w \in \Sigma^*$ is called a terminal word.
- **Intuition:** Production rules map strings with at least one nonterminal to arbitrary other strings.
- **Notation:** If we have n rules $h \rightarrow b_i$ sharing a head, we often write $h \rightarrow b_1 \mid \ldots \mid b_n$ instead.

Phrase Structure Grammars (cont.)

- **Example 2.15.** A simple phrase structure grammar G:

$$\begin{array}{rcl} S & \rightarrow & NP\ Vi \\ NP & \rightarrow & Article\ N \\ Article & \rightarrow & \text{the} \mid \text{a} \mid \text{an} \\ N & \rightarrow & \text{dog} \mid \text{teacher} \mid \ldots \\ Vi & \rightarrow & \text{sleeps} \mid \text{smells} \mid \ldots \end{array}$$

  Here S, is the start symbol, NP, $Article$, N, and Vi are nonterminals.

- **Definition 2.16.** A production rule whose head is a single non-terminal and whose body consists of a single terminal is called lexical or a lexical insertion rule.
  Definition 2.17. The subset of lexical rules of a grammar G is called the lexicon of G and the set of body symbols the vocabulary (or alphabet). The nonterminals in their heads are called lexical categories of G.

- **Definition 2.18.** The non-lexicon production rules are called structural, and the nonterminals in the heads are called phrasal or syntactic categories.

Phrase Structure Grammars (Theory)

- **Idea:** Each symbol sequence in a formal language can be analyzed/generated by the grammar.
- **Definition 2.19.** Given a phrase structure grammar $G := \langle N, \Sigma, P, S \rangle$, we say G derives $t \in (\Sigma \cup N)^*$ from $s \in (\Sigma \cup N)^*$ in one step, iff there is a production rule $p \in P$ with $p = h \rightarrow b$ and there are $u, v \in (\Sigma \cup N)^*$, such that $s = suhv$ and $t = ubv$. We write $s \rightarrow_G^p t$ (or $s \rightarrow_G t$ if p is clear from the context) and use $\rightarrow_G^*$ for the reflexive transitive closure of $\rightarrow_G$. We call $s \rightarrow^*_G t$ a G derivation of t from s.

TEST1:
$$\begin{array}{lll} A & \rightarrow_G & B \\ C & \rightarrow_G & D \end{array}$$

TEST2:
$$\begin{array}{lll} A & \rightarrow_G & B \\ & \rightarrow_G & C \\ & \rightarrow_G & D \end{array}$$

TEST3:
$$\begin{array}{lll} s & \rightarrow_{G_2} & asb \\ & \rightarrow_{G_2} & aaSbb \\ & \rightarrow_{G_2} & aaaSbbb \\ & \rightarrow_{G_2} & aaaaSbbbb \\ & \rightarrow_{G_2} & aaaabbbb \end{array}$$

- **Definition 2.20.** Given a phrase structure grammar $G := \langle N, \Sigma, P, S \rangle$, we say that $s \in (N \cup \Sigma)^*$ is a sentential form of G, iff $S \rightarrow^*_G s$. A sentential form that does not contain nontermials is called a sentence of G, we also say that G accepts s. We say that G rejects s, iff it is not a sentence of G.
- **Definition 2.21.** The language $L(G)$ of G is the set of its sentences. We say that $L(G)$ is generated by G.

Phrase Structure Grammars (Example)

- **Example 2.25.** In the grammar G from ??:

1. *Article* teacher *Vi* is a sentential form,

$$\begin{array}{lll} S & \rightarrow_G & NP\ Vi \\ & \rightarrow_G & Article\ N\ Vi \\ & \rightarrow_G & Article\ \text{teacher}\ Vi \end{array}$$

2. *The teacher sleeps* is a sentence.

$$\begin{array}{lll} S & \rightarrow_G^* & Article\ \text{teacher}\ Vi \\ & \rightarrow_G & \text{the teacher}\ Vi \\ & \rightarrow_G & \text{the teacher sleeps} \end{array}$$

$$\begin{array}{rcl} S & \rightarrow & NP\ Vi \\ NP & \rightarrow & Article\ N \\ Article & \rightarrow & \text{the} \mid \text{a} \mid \text{an} \mid \ldots \\ N & \rightarrow & \text{dog} \mid \text{teacher} \mid \ldots \\ Vi & \rightarrow & \text{sleeps} \mid \text{smells} \mid \ldots \end{array}$$

Grammar Types (Chomsky Hierarchy [Cho65])

- **Observation:** The shape of the grammar determines the "size" of its language.
- **Definition 2.26.** We call a grammar:
  1. context-sensitive (or type 1), if the bodies of production rules have no less symbols than the heads,
  2. context-free (or type 2), if the heads have exactly one symbol,
  3. regular (or type 3), if additionally the bodies are empty or consist of a nonterminal, optionally followed by a terminal symbol.

  By extension, a formal language L is called context-sensitive/context-free/regular (or type 1/type 2/type 3 respectively), iff it is the language of a respective grammar. Context-free grammars are sometimes CFGs and context-free languages CFLs.

Grammar Types (Chomsky Hierarchy [Cho65])

- **Observation:** The shape of the grammar determines the “size” of its language.
- **Definition 2.30.** We call a grammar:
  1. context-sensitive (or type 1), if the bodies of production rules have no less symbols than the heads,
  2. context-free (or type 2), if the heads have exactly one symbol,
  3. regular (or type 3), if additionally the bodies are empty or consist of a nonterminal, optionally followed by a terminal symbol.

  By extension, a formal language L is called context-sensitive/context-free/regular (or type 1/type 2/type 3 respectively), iff it is the language of a respective grammar. Context-free grammars are sometimes CFGs and context-free languages CFLs.
- **Example 2.31 (Context-sensitive).** The language $\{a^{[n]}b^{[n]}c^{[n]}\}$ is accepted by

$$
\begin{array}{rcl}
S & \rightarrow & \mathrm{a\ b\ c} \mid A \\
A & \rightarrow & \mathrm{a}\ A\ B\ \mathrm{c} \mid \mathrm{a\ b\ c} \\
\mathrm{c}\ B & \rightarrow & B\ \mathrm{c} \\
\mathrm{b}\ B & \rightarrow & \mathrm{b\ b}
\end{array}
$$

Grammar Types (Chomsky Hierarchy [Cho65])

- **Observation:** The shape of the grammar determines the "size" of its language.
- **Definition 2.34.** We call a grammar:
  1. context-sensitive (or type 1), if the bodies of production rules have no less symbols than the heads,
  2. context-free (or type 2), if the heads have exactly one symbol,
  3. regular (or type 3), if additionally the bodies are empty or consist of a nonterminal, optionally followed by a terminal symbol.

  By extension, a formal language L is called context-sensitive/context-free/regular (or type 1/type 2/type 3 respectively), iff it is the language of a respective grammar. Context-free grammars are sometimes CFGs and context-free languages CFLs.
- **Example 2.35 (Context-sensitive).** The language $\{a^{[n]}b^{[n]}c^{[n]}\}$
- **Example 2.36 (Context-free).** The language $\{a^{[n]}b^{[n]}\}$ is accepted by $S \to \mathrm{a}\ S\ \mathrm{b} \mid \epsilon$.

Grammar Types (Chomsky Hierarchy [Cho65])

- **Observation:** The shape of the grammar determines the "size" of its language.
- **Definition 2.38.** We call a grammar:
  1. context-sensitive (or type 1), if the bodies of production rules have no less symbols than the heads,
  2. context-free (or type 2), if the heads have exactly one symbol,
  3. regular (or type 3), if additionally the bodies are empty or consist of a nonterminal, optionally followed by a terminal symbol.

  By extension, a formal language L is called context-sensitive/context-free/regular (or type 1/type 2/type 3 respectively), iff it is the language of a respective grammar. Context-free grammars are sometimes CFGs and context-free languages CFLs.
- **Example 2.39 (Context-sensitive).** The language $\{a^{[n]}b^{[n]}c^{[n]}\}$
- **Example 2.40 (Context-free).** The language $\{a^{[n]}b^{[n]}\}$
- **Example 2.41 (Regular).** The language $\{a^{[n]}\}$ is accepted by $S \rightarrow S$ a

Grammar Types (Chomsky Hierarchy [Cho65])

- **Observation:** The shape of the grammar determines the "size" of its language.
- **Definition 2.42.** We call a grammar:
  1. context-sensitive (or type 1), if the bodies of production rules have no less symbols than the heads,
  2. context-free (or type 2), if the heads have exactly one symbol,
  3. regular (or type 3), if additionally the bodies are empty or consist of a nonterminal, optionally followed by a terminal symbol.

  By extension, a formal language L is called context-sensitive/context-free/regular (or type 1/type 2/type 3 respectively), iff it is the language of a respective grammar. Context-free grammars are sometimes CFGs and context-free languages CFLs.
- **Example 2.43 (Context-sensitive).** The language $\{a^{[n]}b^{[n]}c^{[n]}\}$
- **Example 2.44 (Context-free).** The language $\{a^{[n]}b^{[n]}\}$
- **Example 2.45 (Regular).** The language $\{a^{[n]}\}$
- **Observation:** Natural languages are probably context-sensitive but parsable in real time! (like languages low in the hierarchy)

Useful Extensions of Phrase Structure Grammars

- **Definition 2.46.** The Bachus Naur form or Backus normal form (BNF) is a metasyntax notation for context-free grammars.
  It extends the body of a production rule by mutiple (admissible) constructors:
  - alternative: $s_1 \mid \ldots \mid s_n$,
  - repetition: s^* (arbitrary many s) and s^+ (at least one s),
  - optional: $[s]$ (zero or one times),
  - grouping: $(s_1 ; \ldots ; s_n)$, useful e.g. for repetition,
  - character sets: $[s{-}t]$ (all characters c with $s \leq c \leq t$ for a given ordering on the characters), and
  - complements: $[^\wedge s_1, \ldots, s_n]$, provided that the base alphabet is finite.
- **Observation:** All of these can be eliminated, .e.g ($\leadsto$ many more rules)
  - replace $X \to Z\ (s^*)\ W$ with the production rules $X \to Z\ Y\ W$, $Y \to \epsilon$, and $Y \to Y\ s$.
  - replace $X \to Z\ (s^+)\ W$ with the production rules $X \to Z\ Y\ W$, $Y \to s$, and $Y \to Y\ s$.

An Grammar Notation for AI-1

- **Problem:** In grammars, notations for nonterminal symbols should be
  - short and mnemonic (for the use in the body)
  - close to the official name of the syntactic category (for the use in the head)
- In AI-1 we will only use context-free grammars (simpler, but problem still applies)
- **in AI-1:** I will try to give "grammar overviews" that combine those, e.g. the grammar of first-order logic.

variables	X	$\in$	$\mathcal{V}_1$	
function constants	f^k	$\in$	Σ_k^f	
predicate constants	p^k	$\in$	$\Sigma^p{}_k$	
terms	t	$::=$	X	variable
		$\mid$	f^0	constant
		$\mid$	$f^k(t_1, \ldots, t_k)$	application
formulae	A	$::=$	$p^k(t_1, \ldots, t_k)$	atomic
		$\mid$	$\neg\mathrm{A}$	negation
		$\mid$	$\mathrm{A}_1 \wedge \mathrm{A}_2$	conjunction
		$\mid$	$\forall X.\mathrm{A}$	quantifier

4.3 Mathematical Language Recap

Mathematical Structures

- **Observation:** Mathematicians often cast classes of complex objects as mathematical structures.
- We have just seen an example of a mathematical structure: (repeated here for convenience)
- **Definition 3.1.** A phrase structure grammar (also called type 0 grammar, unrestricted grammar, or just grammar) is a tuple $\langle N, \Sigma, P, S\rangle$ where
  - N is a finite set of nonterminal symbols,
  - Σ is a finite set of terminal symbols, members of $\Sigma \cup N$ are called symbols.
  - P is a finite set of production rules: pairs $p := h \rightarrow b$ (also written as $h \Rightarrow b$), where $h \in (\Sigma \cup N)^* N (\Sigma \cup N)^*$ and $b \in (\Sigma \cup N)^*$. The string h is called the head of p and b the body.
  - $S \in N$ is a distinguished symbol called the start symbol (also sentence symbol).

  The sets N and Σ are assumed to be disjoint. Any word $w \in \Sigma^*$ is called a terminal word.
- **Intuition:** All grammars share structure: they have four components, which again share strucct ure, which is further described in the definition above.
- **Observation:** Even though we call production rules "pairs" above, they are also mathematical structures $\langle h, b\rangle$ with a funny notation $h \rightarrow b$.

Mathematical Structures in Programming

- **Observation:** Most programming languages have some way of creating "named structures". Referencing components is usually done via "dot notation".

Mathematical Structures in Programming

- **Observation:** Most programming languages have some way of creating "named structures". Referencing components is usually done via "dot notation".
- **Example 3.4 (Structs in C).** C data structures for representing grammars:

```
struct grule {
  char[][] head;
  char[][] body;
}
struct grammar {
  char[][] nterminals;
  char[][] termininals;
  grule[] grules;
  char[] start;
}
int main() {
  struct grule r1;
  r1.head = "foo";
  r1.body = "bar";
}
```

Mathematical Structures in Programming

- **Observation:** Most programming languages have some way of creating "named structures". Referencing components is usually done via "dot notation".
- **Example 3.6 (Structs in `C`).** `C` data structures for representing grammars:

```
struct grule {
  char[][] head;
  char[][] body;
}
struct grammar {
  char[][] nterminals;
  char[][] termininals;
  grule[] grules;
  char[] start;
}
int main() {
  struct grule r1;
  r1.head = "foo";
  r1.body = "bar";
}
```

- **Example 3.7 (Classes in OOP).** Classes in object-oriented programming languages are based on the same ideas as mathematical structures, only that OOP adds powerful inheritance mechanisms.

In AI-1 we use a mixture between Math and Programming Styles

- In AI-1 we use mathematical notation, ...
- **Definition 3.8.** A structure signature combines the components, their "types", and accessor names of a mathematical structure in a tabular overview.
- **Example 3.9.**

$$\text{grammar} = \left\langle \begin{array}{lll} N & \mathrm{Set} & \text{nonterminal symbols}, \\ \Sigma & \mathrm{Set} & \text{terminal symbols}, \\ P & \{h \rightarrow b \mid \ldots\} & \text{production rules}, \\ S & N & \text{start symbol} \end{array} \right\rangle$$

$$\text{production rule} \quad h \rightarrow b = \left\langle \begin{array}{lll} h & (\Sigma \cup N)^*, N, (\Sigma \cup N)^* & \text{head}, \\ b & (\Sigma \cup N)^* & \text{body} \end{array} \right\rangle$$

  Read the first line "N Set nonterminal symbols" in the structure above as "N is in an (unspecified) set and is a nonterminal symbol".
  Here – and in the future – we will use Set for the class of sets $\leadsto$ "N is a set".
- I will try to give structure signatures where necessary.

Chapter 5
Rational Agents: a Unifying Framework for Artificial Intelligence

5.1 Introduction: Rationality in Artificial Intelligence

What is AI? Going into Details

- **Recap:** AI studies how we can make the computer do things that humans can still do better at the moment. (humans are proud to be rational)
- **What is AI?:** Four possible answers/facets: Systems that

think like humans	think rationally
act like humans	act rationally

expressed by four different definitions/quotes:

	Humanly	**Rational**
Thinking	"*The exciting new effort to make computers think … machines with human-like minds*" [Hau85]	"*The formalization of mental faculties in terms of computational models*" [CM85]
Acting	"*The art of creating machines that perform actions requiring intelligence when performed by people*" [Kur90]	"*The branch of CS concerned with the automation of appropriate behavior in complex situations*" [LS93]

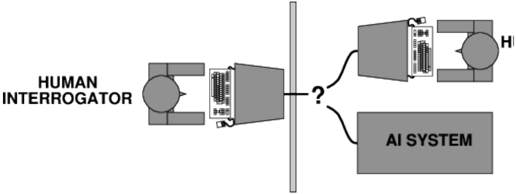
- **Idea:** Rationality is performance-oriented rather than based on imitation.

So, what does modern AI do?

- **Acting Humanly:** Turing test, not much pursued outside Loebner prize
  - $\widehat{=}$ building pigeons that can fly so much like real pigeons that they can fool pigeons
  - Not reproducible, not amenable to mathematical analysis
- **Thinking Humanly:** $\leadsto$ Cognitive Science.
  - How do humans think? How does the (human) brain work?
  - Neural networks are a (extremely simple so far) approximation
- **Thinking Rationally:** Logics, Formalization of knowledge and inference
  - You know the basics, we do some more, fairly widespread in modern AI
- **Acting Rationally:** How to make good action choices?
  - Contains logics (one possible way to make intelligent decisions)
  - We are interested in making good choices in practice (e.g. in AlphaGo)

Acting humanly: The Turing test

- Introduced by Alan Turing (1950) "Computing machinery and intelligence" [Tur50]:
- "Can machines think?" $\longrightarrow$ "Can machines behave intelligently?"
- **Definition 1.1.** The Turing test is an operational test for intelligent behavior based on an imitation game over teletext (arbitrary topic)



- It was predicted that by 2000, a machine might have a 30% chance of fooling a lay person for 5 minutes.
- **Note:** In [Tur50], Alan Turing
  - anticipated all major arguments against AI in following 50 years and
  - suggested major components of AI: knowledge, reasoning, language understanding, learning
- **Problem:** Turing test is not reproducible, constructive, or amenable to mathematical analysis!

Thinking humanly: Cognitive Science

- **1960s:** "cognitive revolution": information processing psychology replaced prevailing orthodoxy of behaviorism.
- Requires scientific theories of internal activities of the brain
- What level of abstraction? "Knowledge" or "circuits"?
- **How to validate?:** Requires
  1. Predicting and testing behavior of human subjects or (top-down)
  2. Direct identification from neurological data. (bottom-up)
- **Definition 1.2.** Cognitive science is the interdisciplinary, scientific study of the mind and its processes. It examines the nature, the tasks, and the functions of cognition.
- **Definition 1.3.** Cognitive neuroscience studies the biological processes and aspects that underlie cognition, with a specific focus on the neural connections in the brain which are involved in mental processes.
- Both approaches/disciplines are now distinct from AI.
- Both share with AI the following characteristic: *the available theories do not explain (or engender) anything resembling human-level general intelligence*
- Hence, all three fields share one principal direction!

Thinking rationally: Laws of Thought

- Normative (or prescriptive) rather than descriptive
- Aristotle: what are correct arguments/thought processes?
- Several Greek schools developed various forms of logic: *notation* and *rules of derivation* for thoughts; may or may not have proceeded to the idea of mechanization.
- Direct line through mathematics and philosophy to modern AI
- **Problems:**
  1. Not all intelligent behavior is mediated by logical deliberation
  2. What is the purpose of thinking? What thoughts *should* I have out of all the thoughts (logical or otherwise) that I *could* have?

Acting Rationally

- **Idea:** Rational behavior $\widehat{=}$ doing the right thing!
- **Definition 1.4.** Rational behavior consists of always doing what is expected to maximize goal achievement given the available information.
- Rational behavior does not necessarily involve thinking e.g., blinking reflex — but thinking should be in the service of rational action.
- **Aristotle:** *Every art and every inquiry, and similarly every action and pursuit, is thought to aim at some good.* (Nicomachean Ethics)

The Rational Agents

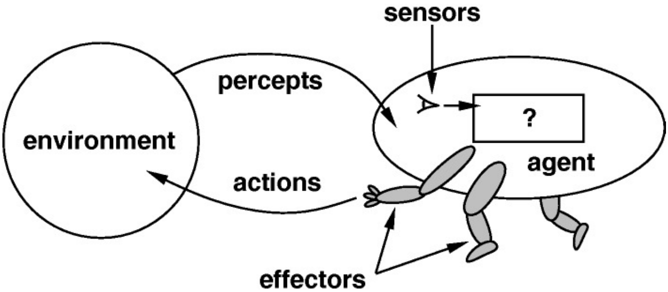
- **Definition 1.5.** An agent is an entity that perceives and acts.
- **Central Idea:** This course is about designing agent that exhibit rational behavior, i.e. for any given class of environments and tasks, we seek the agent (or class of agents) with the best performance.
- **Caveat:** *Computational limitations make perfect rationality unachievable*
  $\rightsquigarrow$ design best program for given machine resources.

5.2 Agents and Environments as a Framework for AI

Agents and Environments

- **Definition 2.1.** An agent is anything that
  - perceives its environment via sensors (a means of sensing the environment)
  - acts on it with actuators (means of changing the environment).

  Definition 2.2. Any recognizable, coherent employment of the actuators of an agent is called an action.



- **Example 2.3.** Agents include humans, robots, softbots, thermostats, etc.
- **remark:** The notion of an agent and its environment is intentionally designed to be inclusive. We will classify and discuss subclasses of both later

Modeling Agents Mathematically and Computationally

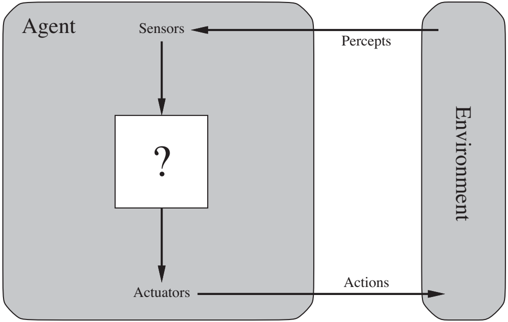
- **Definition 2.4.** A percept is the perceptual input of an agent at a specific time instant.
- **Definition 2.5.** Any recognizable, coherent employment of the actuators of an agent is called an action.
- **Definition 2.6.** The agent function f_a of an agent a maps from percept histories to actions:

$$f_a \colon \mathcal{P}^* \to \mathcal{A}$$

- We assume that agents can always perceive their own actions. (but not necessarily their consequences)
- **Problem:** Agent functions can become very big and may be uncomputable. (theoretical tool only)
- **Definition 2.7.** An agent function can be implemented by an agent program that runs on a (physical or hypothetical) agent architecture.

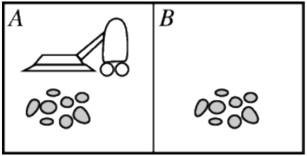
Agent Schema: Visualizing the Internal Agent Structure

- **Agent Schema:** We will use the following kind of agent schema to visualize the internal structure of an agent:



Different agents differ on the contents of the white box in the center.

Example: Vacuum-Cleaner World and Agent



- percepts: location and contents, e.g., $[A, Dirty]$
- actions: *Left*, *Right*, *Suck*, *NoOp*

Percept sequence	Action
$[A, Clean]$	*Right*
$[A, Dirty]$	*Suck*
$[B, Clean]$	*Left*
$[B, Dirty]$	*Suck*
$[A, Clean]$, $[A, Clean]$	*Right*
$[A, Clean]$, $[A, Dirty]$	*Suck*
$[A, Clean]$, $[B, Clean]$	*Left*
$[A, Clean]$, $[B, Dirty]$	*Suck*
$[A, Dirty]$, $[A, Clean]$	*Right*
$[A, Dirty]$, $[A, Dirty]$	*Suck*
$\vdots$	$\vdots$
$[A, Clean]$, $[A, Clean]$, $[A, Clean]$	*Right*
$[A, Clean]$, $[A, Clean]$, $[A, Dirty]$	*Suck*
$\vdots$	$\vdots$

- **Science Question:** What is the *right* agent function?
- **AI Question:** Is there an agent architecture and agent program that implements it.

Table-Driven Agents

- **Idea:** We can just implement the agent function as a lookup table and lookup actions.
- We can directly implement this:

function Table−Driven−Agent(*percept*) **returns** an action
  persistent *table* /∗ a table of actions indexed by percept sequences ∗/
  var *percepts* /∗ a sequence, initially empty ∗/
  append *percept* **to** the **end** of *percepts*
  action := lookup(*percepts*, *table*)
  return *action*

- **Problem:** Why is this not a good idea?
  - The table is much too large: even with n binary percepts whose order of occurrence does not matter, we have 2^n rows in the table.
  - Who is supposed to write this table anyways, even if it "only" has a million entries?

Example: Vacuum-Cleaner Agent Program

- A much better implementation idea is to trigger actions from specific percepts.
- **Example 2.8 (Agent Program).**

procedure Reflex−Vacuum−Agent [*location*,*status*] **returns** an action
  if *status* = Dirty **then return** Suck
  else if *location* = A **then return** Right
  else if *location* = B **then return** Left

- This is the kind of agent programs we will be looking for in AI-1.

5.3 Good Behavior $\rightsquigarrow$ Rationality

Rationality

- **Idea:** Try to design agents that are successful! (aka. "do the right thing")
- **Problem:** What do we mean by "successful", how do we measure "success"?
- **Definition 3.1.** A performance measure is a function that evaluates a sequence of environments.
- **Example 3.2.** A performance measure for a vacuum cleaner could
  - award one point per "square" cleaned up in time T?
  - award one point per clean "square" per time step, minus one per move?
  - penalize for $> k$ dirty squares?
- **Definition 3.3.** An agent is called rational, if it chooses whichever action maximizes the expected value of the performance measure given the percept sequence to date.
- **Critical Observation:** We only need to maximize the expected value, not the actual value of the performance measure!
- **Question:** Why is rationality a good quality to aim for?

Consequences of Rationality: Exploration, Learning, Autonomy

- **Note:** A rational agent need not be perfect:
  - It only needs to maximize expected value (rational $\neq$ omniscient)
    - need not predict e.g. very unlikely but catastrophic events in the future
  - Percepts may not supply all relevant information (rational $\neq$ clairvoyant)
    - if we cannot perceive things we do not need to react to them.
    - but we may need to try to find out about hidden dangers (exploration)
  - Action outcomes may not be as expected (rational $\neq$ successful)
    - but we may need to take action to ensure that they do (more often) (learning)
- **Note:** Rationality may entail exploration, learning, autonomy (depending on the environment / task)
- **Definition 3.4.** An agent is called autonomous, if it does not rely on the prior knowledge about the environment of the designer.
- Autonomy avoids fixed behaviors that can become unsuccessful in a changing environment. (anything else would be irrational)
- The agent may have to learn all relevant traits, invariants, properties of the environment and actions.

PEAS: Describing the Task Environment

- **Observation:** To design a rational agent, we must specify the task environment in terms of performance measure, environment, actuators, and sensors, together called the PEAS components.
- **Example 3.5.** When designing an automated taxi:
  - **Performance measure:** safety, destination, profits, legality, comfort, ...
  - **Environment:** US streets/freeways, traffic, pedestrians, weather, ...
  - **Actuators:** steering, accelerator, brake, horn, speaker/display, ...
  - **Sensors:** video, accelerometers, gauges, engine sensors, keyboard, GPS, ...
- **Example 3.6 (Internet Shopping Agent).** The task environment:
  - Performance measure: price, quality, appropriateness, efficiency
  - Environment: current and future WWW sites, vendors, shippers
  - Actuators: display to user, follow URL, fill in form
  - Sensors: HTML pages (text, graphics, scripts)

Examples of Agents: PEAS descriptions

Agent Type	Performance measure	Environment	Actuators	Sensors
Chess/Go player	win/loose/draw	game board	moves	board position
Medical diagnosis system	accuracy of diagnosis	patient, staff	display questions, diagnoses	keyboard entry of symptoms
Part-picking robot	percentage of parts in correct bins	conveyor belt with parts, bins	jointed arm and hand	camera, joint angle sensors
Refinery controller	purity, yield, safety	refinery, operators	valves, pumps, heaters, displays	temperature, pressure, chemical sensors
Interactive English tutor	student's score on test	set of students, testing accuracy	display exercises, suggestions, corrections	keyboard entry

Agents

- Which are agents?
  (A) James Bond.
  (B) Your dog.
  (C) Vacuum cleaner.
  (D) Thermometer.

Agents

- Which are agents?
  - (A) James Bond.
  - (B) Your dog.
  - (C) Vacuum cleaner.
  - (D) Thermometer.
- **Answer:**
  - (A/B) : Definite yes. (James Bond & your dog)
  - (C) : Yes, if it's an autonomous vacuum cleaner. Else, no.
  - (D) : No, because it cannot do anything. (Changing the displayed temperature value could be considered an "action", but that is not the intended usage of the term)

5.4 Classifying Environments

Environment types

- **Observation 4.1.** *Agent design is largely determined by the type of environment it is intended for.*
- **Problem:** There is a vast number of possible kinds of environments in AI.
- **Solution:** Classify along a few "dimensions". (independent characteristics)
- **Definition 4.2.** For an agent a we classify the environment e of a by its type, which is one of the following. We call e
  1. fully observable, iff the a's sensors give it access to the complete state of the environment at any point in time, else partially observable.
  2. deterministic, iff the next state of the environment is completely determined by the current state and a's action, else stochastic.
  3. episodic, iff a's experience is divided into atomic episodes, where it perceives and then performs a single action. Crucially, the next episode does not depend on previous ones. Non-episodic environments are called sequential.
  4. dynamic, iff the environment can change without an action performed by a, else static. If the environment does not change but a's performance measure does, we call e semidynamic.
  5. discrete, iff the sets of e's state and a's actions are countable, else continuous.
  6. single-agent, iff only a acts on e; else multi-agent(when must we count parts of e as agents?)

Environment Types (Examples)

- **Example 4.3.** Some environments classified:

	Solitaire	Backgammon	Internet shopping	Taxi
fully observable	No	Yes	No	No
deterministic	Yes	No	Partly	No
episodic	No	Yes	No	No
static	Yes	Semi	Semi	No
discrete	Yes	Yes	Yes	No
single-agent	Yes	No	Yes (except auctions)	No

Environment Types (Examples)

- **Example 4.6.** Some environments classified:

	Solitaire	Backgammon	Internet shopping	Taxi
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- **Note:** Take the example above with a grain of salt. There are often multiple interpretations that yield different classifications and different agents. (agent designer's choice)
- **Example 4.7.** Seen as a multi-agent game, chess is deterministic, as a single-agent game, it is stochastic.

Environment Types (Examples)

- **Example 4.9.** Some environments classified:

	Solitaire	Backgammon	Internet shopping	Taxi
fully observable	No	Yes	No	No
deterministic	Yes	No	Partly	No
episodic	No	Yes	No	No
static	Yes	Semi	Semi	No
discrete	Yes	Yes	Yes	No
single-agent	Yes	No	Yes (except auctions)	No

- **Note:** Take the example above with a grain of salt. There are often multiple interpretations that yield different classifications and different agents. (agent designer's choice)
- **Example 4.10.** Seen as a multi-agent game, chess is deterministic, as a single-agent game, it is stochastic.
- **Observation 4.11.** *The real world is (of course) a partially observable, stochastic, sequential, dynamic, continuous, and multi-agent environment.* (*worst case for AI*)
- **Preview:** We will concentrate on the "easy" environment types (fully observable, deterministic, episodic, static, and single-agent) in AI-1 and extend them to "realworld"-compatible ones in AI-2.

5.5 Types of Agents

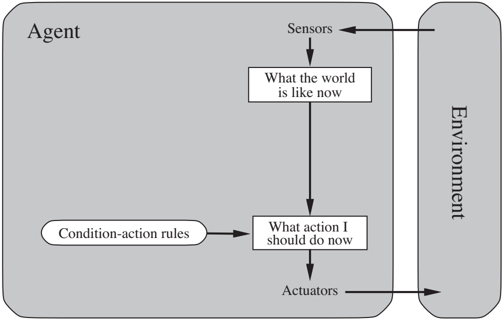
Agent Types

- **Observation:** So far we have described (and analyzed) agents only by their behavior (cf. agent function $f\colon \mathcal{P}^* \to \mathcal{A}$).
- **Problem:** This does not help us to build agents. (the goal of AI)
- To build an agent, we need to fix an agent architecture and come up with an agent program that runs on it.
- **Preview:** Four basic types of agent architectures in order of increasing generality:
  1. simple reflex agents
  2. model-based agents
  3. goal-based agents
  4. utility-based agents

  All these can be turned into learning agents.

Simple reflex agents

- **Definition 5.1.** A simple reflex agent is an agent a that only bases its actions on the last percept: so the agent function simplifies to $f_a : \mathcal{P} \to \mathcal{A}$.
- **Agent Schema:**



- **Example 5.2 (Agent Program).**

procedure Reflex−Vacuum−Agent [location,status] **returns** an action
  if status = Dirty **then** ...

Simple reflex agents (continued)

- **General Agent Program:**

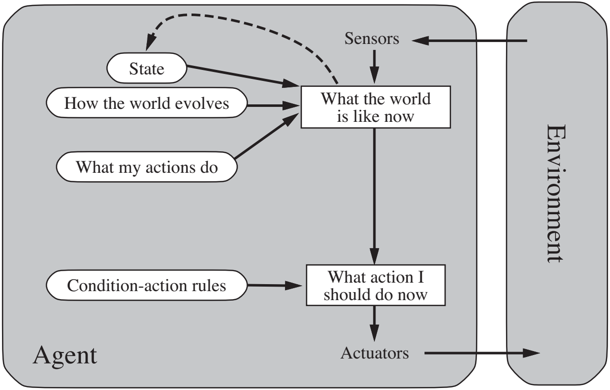
```
function Simple−Reflex−Agent (percept) returns an action
  persistent: rules /* a set of condition−action rules*/

  state := Interpret−Input(percept)
  rule := Rule−Match(state,rules)
  action := Rule−action[rule]
  return action
```

- **Problem:** Simple reflex agents can only react to the perceived state of the environment, not to changes.
- **Example 5.3.** Automobile tail lights signal braking by brightening. A simple reflex agent would have to compare subsequent percepts to realize.
- **Problem:** Partially observable environments get simple reflex agents into trouble.
- **Example 5.4.** Vacuum cleaner robot with defective location sensor $\rightsquigarrow$ infinite loops.

Model-based Reflex Agents: Idea

- **Idea:** Keep track of the state of the world we cannot see in an internal model.
- **Agent Schema:**



Model-based Reflex Agents: Definition

- **Definition 5.5.** A model-based agent is an agent whose actions depend on
  - a world model: a set $\mathcal{S}$ of possible states.
  - a sensor model S that given a state s and a percepts p determines a new state $S(s, p)$.
  - a transition model $\mathcal{T}$, that predicts a new state $\mathcal{T}(s, a)$ from a state s and an action a.
  - An action function f that maps (new) states to an actions.

  If the world model of a model-based agent A is in state s and A has taken action a, A will transition to state $s' = \mathcal{T}(S(p, s), a)$ and take action $a' = f(s')$.
- **Note:** As different percept sequences lead to different states, so the agent function $f_a \colon \mathcal{P}^* \to \mathcal{A}$ no longer depends only on the last percept.
- **Example 5.6 (Tail Lights Again).** Model-based agents can do the **??** if the states include a concept of tail light brightness.

Model-Based Agents (continued)

- **Observation 5.7.** *The agent program for a model-based agent is of the following form:*

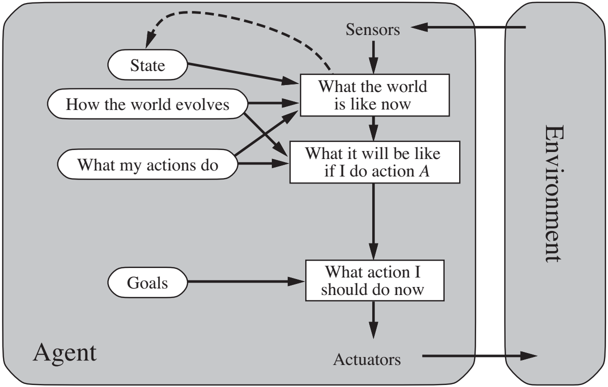
```
function Model−Based−Agent (percept) returns an action
  var state /∗ a description of the current state of the world ∗/
  persistent rules /∗ a set of condition−action rules ∗/
  var action /∗ the most recent action, initially none ∗/

  state := Update−State(state,action,percept)
  rule := Rule−Match(state,rules)
  action := Rule−action(rule)
  return action
```

- **Problem:** Having a world model does not always determine what to do (rationally).
- **Example 5.8.** Coming to an intersection, where the agent has to decide between going left and right.

Goal-based Agents

- **Problem:** A world model does not always determine what to do (rationally).
- **Observation:** Having a goal in mind does! (determines future actions)
- **Agent Schema:**



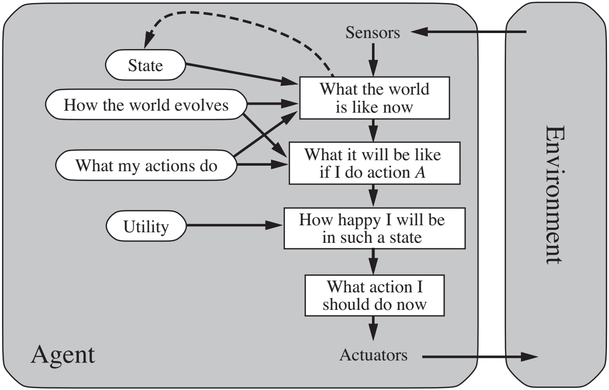
Goal-based agents (continued)

- **Definition 5.9.** A goal-based agent is a model-based agent with transition model T that deliberates actions based on 3 and a world model: It employs
  - a set $\mathcal{G}$ of goals and a goal function f that given a (new) state s' selects an action a to best reach $\mathcal{G}$.

  The action function is then $s \mapsto f(T(s), \mathcal{G})$.
- **Observation:** A goal-based agent is more flexible in the knowledge it can utilize.
- **Example 5.10.** A goal-based agent can easily be changed to go to a new destination, a model-based agent's rules make it go to exactly one destination.

Utility-based Agents

- **Definition 5.11.** A utility-based agent uses a world model along with a utility function that models its preferences among the states of that world. It chooses the action that leads to the best expected utility.
- **Agent Schema:**



Utility-based vs. Goal-based Agents

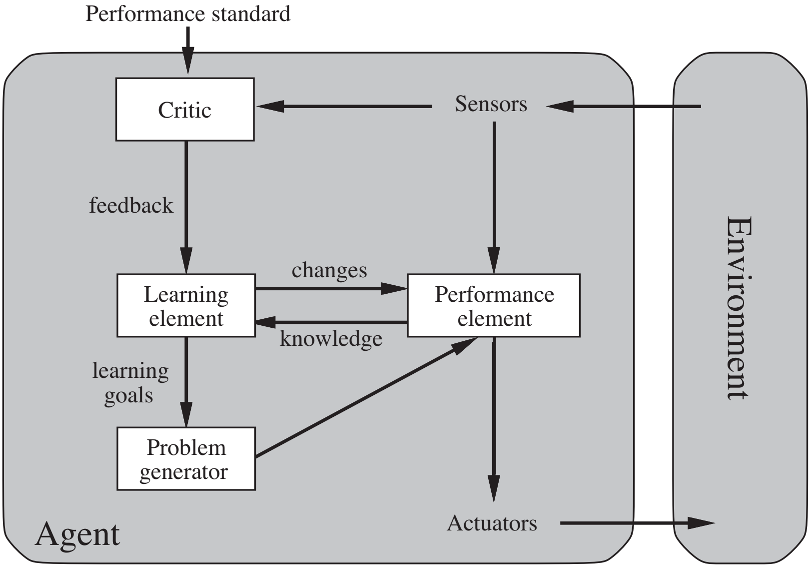
- **Question:** What is the difference between goal-based and utility-based agents?
- **Utility-based Agents are a Generalization:** We can always force goal-directedness by a utility function that only rewards goal states.
- **Goal-based Agents can do less:** A utility function allows rational decisions where mere goals are inadequate:
  - conflicting goals (utility gives tradeoff to make rational decisions)
  - goals obtainable by uncertain actions (utility $\times$ likelihood helps)

Learning Agents

- **Definition 5.12.** A learning agent is an agent that augments the performance element – which determines actions from percept sequences with
  - a learning element which makes improvements to the agent's components,
  - a critic which gives feedback to the learning element based on an external performance standard,
  - a problem generator which suggests actions that lead to new and informative experiences.
- The performance element is what we took for the whole agent above.

Learning Agents

- **Agent Schema:**



Learning Agents: Example

- **Example 5.13 (Learning Taxi Agent).** It has the components
  - Performance element: the knowledge and procedures for selecting driving actions. (this controls the actual driving)
  - critic: observes the world and informs the learning element (e.g. when passengers complain brutal braking)
  - Learning element modifies the braking rules in the performance element (e.g. earlier, softer)
  - Problem generator might experiment with braking on different road surfaces
- The learning element can make changes to any "knowledge components" of the diagram, e.g. in the
  - model from the percept sequence (how the world evolves)
  - success likelihoods by observing action outcomes (what my actions do)
- **Observation:** here, the passenger complaints serve as part of the "external performance standard" since they correlate to the overall outcome – e.g. in form of tips or blacklists.

Domain-Specific vs. General Agents

▶

Domain-Specific Agent	vs.	**General Agent**
 Duell Kasparow gegen Deep Blue (1997): *Demütigende Niederlage*	vs.	
Solver specific to a particular problem (“domain”).	vs.	Solver based on *description* in a general problem-description language (e.g., the rules of any board game).
More efficient.	vs.	Much less design/maintenance work.

▶ What kind of agent are you?

5.6 Representing the Environment in Agents

Representing the Environment in Agents

- We have seen various components of agents that answer questions like
  - *What is the world like now?*
  - *What action should I do now?*
  - *What do my actions do?*
- **Next natural question:** How do these work? (see the rest of the course)

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- **Important Distinction:** How the agent implements the world model.
- **Definition 6.2.** We call a state representation
  - atomic, iff it has no internal structure (black box)
  - factored, iff each state is characterized by attributes and their values.
  - structured, iff the state includes representations of objects, their properties and relationships.

Representing the Environment in Agents

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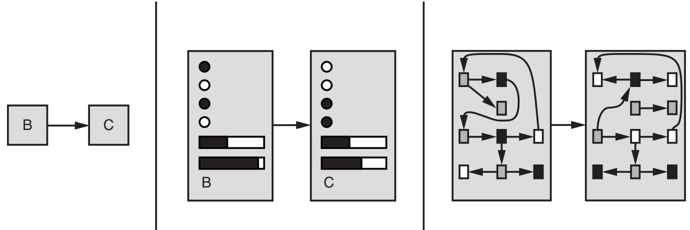
  - structured, iff the state includes representations of objects, their properties and relationships.
- **Intuition:** From atomic to structured, the representations agent designer more flexibility and the algorithms more components to process.
- **Also** The additional internal structure will make the algorithms more complex.

Atomic/Factored/Structured State Representations

- **Schematically:** We can visualize the three kinds by



(a) Atomic (b) Factored (b) Structured

- **Example 6.4.** Consider the problem of finding a driving route from one end of a country to the other via some sequence of cities.
  - In an atomic representation the state is represented by the name of a city.

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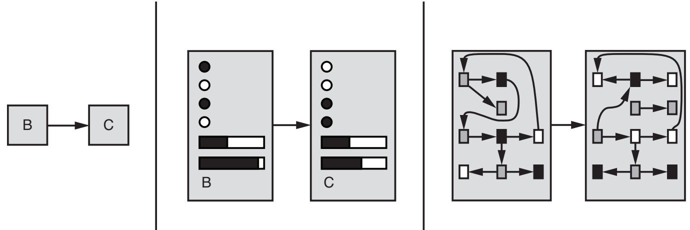

(a) Atomic (b) Factored (b) Structured

- **Example 6.5.** Consider the problem of finding a driving route from one end of a country to the other via some sequence of cities.
  - In an atomic representation the state is represented by the name of a city.
  - In a factored representation we may have attributes “gps-location”, “gas”,... (allows information sharing between states and uncertainty)
  - But how to represent a situation, where a large truck blocking the road, since it is trying to back into a driveway, but a loose cow is blocking its path. (attribute “TruckAheadBackingIntoDairyFarmDrivewayBlockedByLooseCow” is unlikely)

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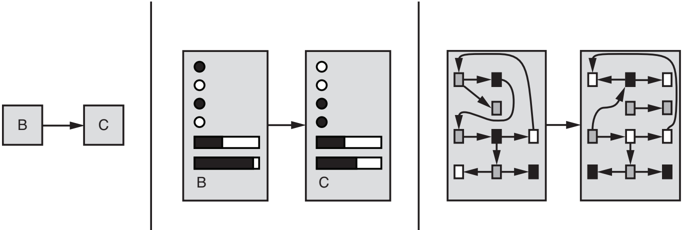

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  - In a structured representation, we can have objects for trucks, cows, etc. and their relationships. (at "run-time")

5.7 Rational Agents: Summary

Summary

- Agents interact with environments through actuators and sensors.
- The agent function describes what the agent does in all circumstances.
- The performance measure evaluates the environment sequence.
- A perfectly rational agent maximizes expected performance.
- Agent programs implement (some) agent functions.
- PEAS descriptions define task environments.
- Environments are categorized along several dimensions:
  fully observable? deterministic? episodic? static? discrete? single-agent?
- Several basic agent architectures exist:
  reflex, model-based, goal-based, utility-based

Corollary: We are Agent Designers!

- **State:** We have seen (and will add more details to) different
  - agent architectures,
  - corresponding agent programs and algorithms, and
  - world representation paradigms.
- **Problem:** Which one is the best?

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- **State:** We have seen (and will add more details to) different
  - agent architectures,
  - corresponding agent programs and algorithms, and
  - world representation paradigms.
- **Problem:** Which one is the best?
- **Answer:** That really depends on the environment type they have to survive/thrive in! The agent designer – i.e. you – has to choose!
  - The course gives you the necessary competencies.
  - There is often more than one reasonable choice.
  - Often we have to build agents and let them compete to see what really works.

- **Consequence:** The rational agents paradigm used in this course challenges you to become a good agent designer.

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