Assignment10 – Inference in First-Order Logic

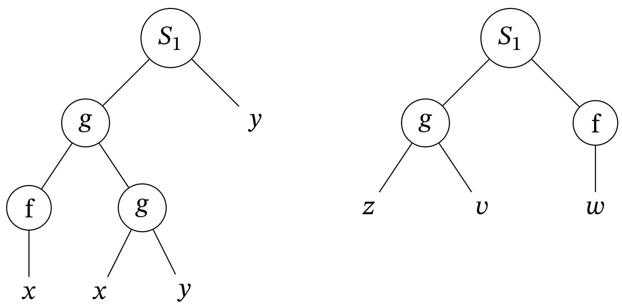
Problem 10.1 (Unification)

Let $S_1 \in \Sigma_2^f, S_2 \in \Sigma_3^f, f \in \Sigma_1^f, g \in \Sigma_2^f, c \in \Sigma_0^f$

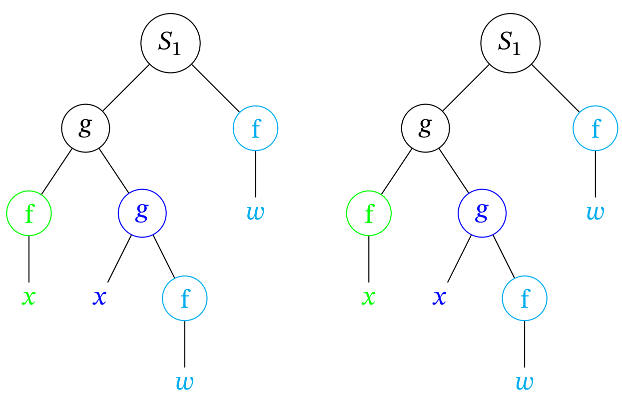
Decide whether (and how or why not) the following pairs of terms are unifiable.

1. $S_1(g(f(x), g(x, y)), y)$ and $S_1(g(z, v), f(w))$

Solution: The term trees look like this:



Obviously, we need to perform the following substitutions to make the two trees equal:



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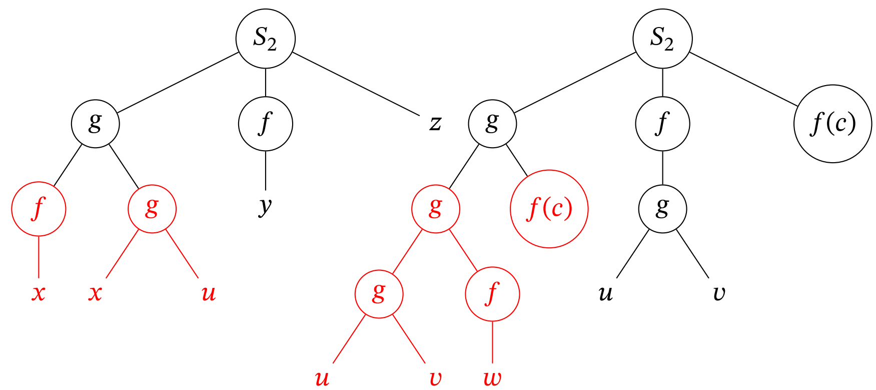
2. $S_2(g(f(x), g(x, u)), f(y), z)$ and $S_2(g(g(g(u, v), f(w)), f(c)), f(g(u, v)), f(c))$

Solution: The term trees look like this:



Obviously, the red subtrees can't be unified.

Problem 10.2 (Natural Deduction)

Let $R \in \Sigma_2^p$, $P \in \Sigma_1^p$, $c \in \Sigma_0^f$.
Prove the following formula in Natural Deduction:

$$(\forall X.\forall Y.R(Y,X) \Rightarrow P(Y)) \wedge (\exists Y.R(c,Y)) \Rightarrow P(c)$$

Solution:

1(Assumption)^1	$(\forall X.\forall Y.R(Y,X) \Rightarrow P(Y), \exists Y.R(c,Y))$
$2\mathcal{ND}_0 \wedge E_l(1)$	$\forall X.\forall Y.R(Y,X) \Rightarrow P(Y)$
$3\mathcal{ND}_0 \wedge E_r(1)$	$\exists Y.R(c,Y)$
$4\mathcal{ND}^1 \forall E(2)$	$\forall Y.R(Y,X) \Rightarrow P(Y)$
$5\mathcal{ND}^1 \forall E(4)$	$R(c,X) \Rightarrow P(c)$
$6\mathcal{ND}^1 \forall I(5)$	$\forall X.R(c,X) \Rightarrow P(c)$
7(Assumption)^2	$R(c,d)$
$8\mathcal{ND}^1 \forall E(6)$	$R(c,d) \Rightarrow P(c)$
$9\mathcal{ND}_0 \Rightarrow E(8,7)$	$P(c)$
$10\mathcal{ND}^1 \exists E^2(3,9)$	$P(c)$
$11\mathcal{ND}_0 \Rightarrow I^1(10)$	$((\forall X.\forall Y.R(Y,X)) \Rightarrow P(Y)) \wedge (\exists Y.R(c,Y)) \Rightarrow P(c)$

Problem 10.3 (First-Order Tableaux)

Prove or refute the following formula using the first-order *free variable tableaux calculus*. We have $P, Q \in \Sigma_1^p$.

$$(\forall X.P(X) \Rightarrow Q(X)) \Rightarrow (((\forall X.P(X)) \Rightarrow (\forall.Q(X))))$$

Solution:

$$
\begin{array}{c}
((\forall X.P(X) \Rightarrow Q(X)) \Rightarrow ((\forall X.P(X)) \Rightarrow (\forall.Q(X))))^{\mathsf{F}} \\
(\forall X.P(X) \Rightarrow Q(X))^{\mathsf{T}} \\
((\forall X.P(X)) \Rightarrow (\forall X.Q(X)))^{\mathsf{F}} \\
(P(Y) \Rightarrow Q(Y))^{\mathsf{T}} \\
(\forall X.P(X))^{\mathsf{T}} \\
(\forall X.Q(X))^{\mathsf{F}} \\
P(Z)^{\mathsf{T}} \\
Q(c)^{\mathsf{F}} \\
\begin{array}{c|c}
P(Y)^{\mathsf{F}} & Q(Y)^{\mathsf{T}} \\
\bot : [Y/Z] & \bot : [c/Y]
\end{array}
\end{array}
$$
