Assignment1 – Probability

Given: May 2   Due: May 6

Problem 1.1 (Simple Sample Spaces)

In many important situations including all problems treated in this *course*, the sample space, probability measure, domains, and random variables can be given in a simplified form, namely by:

- a list of random variable declarations $X_1, \dots, X_n$, each consisting of
  - a name such as X
  - a finite domain such as $D_X = \{0, 1, 2, 3\}$
- a probability function $\Omega \to [0; 1]$ where $\Omega = D_{X_1} \times \dots \times D_{X_n}$ such that $\Sigma_{e\in\Omega} P(e) = 1$

Define the corresponding probability space $\langle \Omega, Q\rangle$ and show that it satisfies the Kolmogorov axioms.
Define the random variables $Y_1, \dots, Y_n$ induced by the respective X_i.

Solution: $Q : \mathcal{P}(\Omega) \to [0; 1]$ is defined by $Q(A) = \Sigma_{e\in A} P(e)$. (Note this is a finite sum because all the D_X and thus Ω and A are finite.)

To show the Kolmogorov axioms:

- $Q(\Omega) = \Sigma_{e\in\Omega} P(e) = 1$
- $Q(\bigcup_i A_i) = \Sigma_{e\in\bigcup_i A_i} P(e) =$ (because the A_i are pairwise disjoint.) $\Sigma_i \Sigma_{e\in A_i} P(e) = \Sigma_i Q(A_i)$.

For each X_i with domain D_{X_i}, we define a random variable $Y_i : \Omega \to D_{X_i}$ by $Y_i(x_1, \dots, x_n) = x_i$ for $(x_1, \dots, x_n) \in \Omega$.

Problem 1.2 (Calculations)

Assume *random variables* X, Y both with *domain* $\{0, 1, 2\}$, whose *joint probability distribution* $P(X, Y)$ is given by

x	y	$P(X = x, Y = y)$
0	0	a
0	1	b
0	2	c
1	0	d
1	1	e
1	2	f
2	0	g
2	1	h
2	2	i

1. Give all *subsets* of the *probabilities* $\{a, b, c, d, e, f, g, h, i\}$ that *sum* to 1.

Solution: Only $\{a, b, c, d, e, f, g, h, i\}$

2. In terms of $a, b, c, d, e, f, g, h, i$, give $P(X \neq 0)$.

Solution: $d + e + f + g + h + i$

3. In terms of $a, b, c, d, e, f, g, h, i$, give $P(X = 1, Y = 0)$.

Solution: d

4. In terms of $a, b, c, d, e, f, g, h, i$, give $P(X = 1 \mid Y = 0)$.

Solution: $d/(a + d + g)$

5. In terms of $a, b, c, d, e, f, g, h, i$, give $P(X + Y = 2)$.

Solution: $c + e + g$

6. In terms of $a, b, c, d, e, f, g, h, i$, give $P(X + Y = 2 \mid X > Y)$.

Solution: $g/(d + g + h)$

Problem 1.3 (Stochastic and Conditional independence)

Consider the following random variables:

- three flips C_1, C_2, and C_3 of the same fair coin, which can be heads or tails
- the variable E which is 1 if both C_1 and C_2 are heads and 0 otherwise
- the variable F which is 1 if both C_2 and C_3 are heads and 0 otherwise

Out of the above 5 random variables,

1. Give three random variables X, Y, Z such that X and Y are stochastically independent but not conditionally independent given Z,

Solution: E.g., C_1 and C_2 with $Z = E$.

2. Give three random variables X, Y, Z such that X and Y are not stochastically independent but conditionally independent given Z.

Solution: E.g., E and F with $Z = C_2$.

Problem 1.4 (Basic Probability)

Let A, B, C be Boolean random variables, and let a, b, c denote the atomic events that A, B, C, respectively, are true. Which of the following equalities are always true? Justify each of your answers in one sentence.

1. $P(b) = P(a,b) + P(\neg a, b)$

Solution: True (marginalization over A)

2. $P(a) = P(a \mid b) + P(a \mid \neg b)$

Solution: Not true (e.g. $P(a \mid b) = P(a \mid \neg b) = 0.6$ would result in $P(a) = 1.2$)

3. $P(a,b) = P(a) \cdot P(b)$

Solution: Not true (only true if A and B are stochastically independent)

4. $P(a,b \mid c) \cdot P(c) = P(c,a \mid b) \cdot P(b)$

Solution: True (using product rule, both sides become $P(a,b,c)$)

5. $P(a \vee b) = P(a) + P(b)$

Solution: Not true (general form is $P(a \vee b) = P(a) + P(b) - P(a,b)$)

6. $P(a, \neg b) = (1 - P(b \mid a)) \cdot P(a)$

Solution: True ($1 - P(b \mid a) = P(\neg b \mid a)$ and via product rule we get $P(a, \neg b)$)

Problem 1.5 (Chained Production Elements)

An apparatus consists of six elements A, B, C, D, E, F. Assume the probabilities $P(b_X)$, that element X breaks down, are all stochastically independent, with $P(b_A) = 5\%$, $P(b_B) = 10\%$, $P(b_C) = 15\%$, $P(b_D) = 20\%$, $P(b_E) = 25\%$, and $P(b_F) = 30\%$.

Note: We deliberately differentiate between *not being operational* and *being broken*. If an element breaks, it is not operational; if an element is not operational, either it or the linked element broke.

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1. Assume the apparatus works if and only if at least A and B are operational, C and D are operational, or E and F are operational. What is the probability the apparatus works?

Solution: Let W be a random variable stating that the apparatus works. Let O_X be a random variable indicating that element X is operational.
In this problem, o_X is equivalent to $\neg b_X$ for all elements X.

$$\begin{aligned}
P(w) &= P(o_A \wedge o_B \vee o_C \wedge o_D \vee o_E \wedge o_F) \\
&= 1 - P(\underbrace{\neg(o_A \wedge o_B) \wedge \neg(o_C \wedge o_D) \wedge \neg(o_E \wedge o_F)}_{\text{all events are independent}}) \\
&= 1 - P(\neg(o_A \wedge o_B)) \cdot P(\neg(o_C \wedge o_D)) \cdot P(\neg(o_E \wedge o_F)) \\
&= 1 - P(\neg o_A \vee \neg o_B) \cdot P(\neg o_C \vee \neg o_D) \cdot P(\neg o_E \vee \neg o_F) \\
&= 1 - P(b_A \vee b_B) \cdot P(b_C \vee b_D) \cdot P(b_E \vee b_F) \\
&= 1 - (P(b_A) + P(b_B) - P(b_A) \cdot P(b_B)) \cdot (P(b_C) + P(b_D) - P(b_C) \cdot P(b_D)) \\
&\quad \cdot (P(b_E) + P(b_F) - P(b_E) \cdot P(b_F)) \\
&= 1 - (0.05 + 0.1 - (0.05 \cdot 0.1)) \cdot (0.15 + 0.2 - (0.15 \cdot 0.2)) \cdot (0.25 + 0.3 - (0.25 \cdot 0.3))
\end{aligned}$$

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2. Consider a different scenario, in which the elements A and C, D and F and B and E are pairwise linked; such that if either of them breaks down, then the linked element is not operational either. What is the probability that the apparatus works now?

Solution: Using the exclusion principle:

$$\begin{aligned}
P(w) &= P(o_A \wedge o_B \vee o_C \wedge o_D \vee o_E \wedge o_F) \\
&= P(o_A, o_B) + P(o_C, o_D) + P(o_E, o_F) - P(o_A, o_B, o_C, o_D) - P(o_A, o_B, o_E, o_F) - P(o_C, o_D, o_E, o_F) \\
&\quad + P(o_A, o_B, o_C, o_D, o_E, o_F)
\end{aligned}$$

Due to the links, o_X is equivalent to $\neg b_X \wedge \neg b_{\widetilde{X}}$ where $\widetilde{X}$ is the element that X is linked with. So, for example, o_A is equivalent to $\neg b_A \wedge \neg b_C$. This gives us

$$\begin{aligned}
P(w) = {} & \underbrace{P(\neg b_A, \neg b_C, \neg b_B, \neg b_E)}_{=P(o_A, o_B)} + P(\neg b_A, \neg b_C, \neg b_D, \neg b_F) + P(\neg b_B, \neg b_E, \neg b_D, \neg b_F) \\
& - P(\neg b_A, \neg b_B, \neg b_C, \neg b_D, \neg b_E, \neg b_F) - P(\neg b_A, \neg b_B, \neg b_C, \neg b_D, \neg b_E, \neg b_F) \\
& - P(\neg b_A, \neg b_B, \neg b_C, \neg b_D, \neg b_E, \neg b_F) + P(\neg b_A, \neg b_B, \neg b_C, \neg b_D, \neg b_E, \neg b_F) \\
= {} & P(\neg b_A, \neg b_C, \neg b_B, \neg b_E) + P(\neg b_A, \neg b_C, \neg b_D, \neg b_F) + P(\neg b_B, \neg b_E, \neg b_D, \neg b_F) \\
& - 2P(\neg b_A, \neg b_B, \neg b_C, \neg b_D, \neg b_E, \neg b_F) \\
= {} & P(\neg b_A)P(\neg b_B)P(\neg b_C)P(\neg b_E) + P(\neg b_A)P(\neg b_C)P(\neg b_D)P(\neg b_F) + P(\neg b_B)P(\neg b_D)P(\neg b_E)P(\neg b_F) \\
& - 2P(\neg b_A)P(\neg b_B)P(\neg b_C)P(\neg b_D)P(\neg b_E)P(\neg b_F) \\
= {} & 0.5450625 + 0.4522 + 0.378 - 2 \cdot 0.305235 \approx 76\%
\end{aligned}$$
