Assignment1 – Probability

Given: May 2   Due: May 6

Problem 1.1 (Simple Sample Spaces)

In many important situations including all problems treated in this *course*, the sample space, probability measure, domains, and random variables can be given in a simplified form, namely by:

- a list of random variable declarations $X_1, \ldots, X_n$, each consisting of
  - a name such as X
  - a finite domain such as $D_X = \{0, 1, 2, 3\}$
- a probability function $\Omega \to [0; 1]$ where $\Omega = D_{X_1} \times \ldots \times D_{X_n}$ such that $\Sigma_{e \in \Omega} P(e) = 1$

Define the corresponding probability space $\langle \Omega, Q \rangle$ and show that it satisfies the Kolmogorov axioms.
Define the random variables $Y_1, \ldots, Y_n$ induced by the respective X_i.

Problem 1.2 (Calculations)

Assume *random variables* X, Y both with *domain* $\{0, 1, 2\}$, whose *joint probability distribution* $P(X, Y)$ is given by

x	y	$P(X = x, Y = y)$
0	0	a
0	1	b
0	2	c
1	0	d
1	1	e
1	2	f
2	0	g
2	1	h
2	2	i

1. Give all *subsets* of the *probabilities* $\{a, b, c, d, e, f, g, h, i\}$ that *sum* to 1.
2. In terms of $a, b, c, d, e, f, g, h, i$, give $P(X \neq 0)$.
3. In terms of $a, b, c, d, e, f, g, h, i$, give $P(X = 1, Y = 0)$.
4. In terms of $a, b, c, d, e, f, g, h, i$, give $P(X = 1 \mid Y = 0)$.
5. In terms of $a, b, c, d, e, f, g, h, i$, give $P(X + Y = 2)$.
6. In terms of $a, b, c, d, e, f, g, h, i$, give $P(X + Y = 2 \mid X > Y)$.

Problem 1.3 (Stochastic and Conditional independence)

Consider the following random variables:

- three flips C_1, C_2, and C_3 of the same fair coin, which can be heads or tails
- the variable E which is 1 if both C_1 and C_2 are heads and 0 otherwise

- the variable F which is 1 if both C_2 and C_3 are heads and 0 otherwise

Out of the above 5 random variables,

1. Give three random variables X, Y, Z such that X and Y are stochastically independent but not conditionally independent given Z,
2. Give three random variables X, Y, Z such that X and Y are not stochastically independent but conditionally independent given Z.

Problem 1.4 (Basic Probability)

Let A, B, C be Boolean random variables, and let a, b, c denote the atomic events that A, B, C, respectively, are true. Which of the following equalities are always true? Justify each of your answers in one sentence.

1. $P(b) = P(a, b) + P(\neg a, b)$
2. $P(a) = P(a \mid b) + P(a \mid \neg b)$
3. $P(a, b) = P(a) \cdot P(b)$
4. $P(a, b \mid c) \cdot P(c) = P(c, a \mid b) \cdot P(b)$
5. $P(a \vee b) = P(a) + P(b)$
6. $P(a, \neg b) = (1 - P(b \mid a)) \cdot P(a)$

Problem 1.5 (Chained Production Elements)

An apparatus consists of six elements A, B, C, D, E, F. Assume the probabilities $P(b_X)$, that element X breaks down, are all stochastically independent, with $P(b_A) = 5\%$, $P(b_B) = 10\%$, $P(b_C) = 15\%$, $P(b_D) = 20\%$, $P(b_E) = 25\%$, and $P(b_F) = 30\%$.

Note: We deliberately differentiate between *not being operational* and *being broken*. If an element breaks, it is not operational; if an element is not operational, either it or the linked element broke.

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1. Assume the apparatus works if and only if at least A and B are operational, C and D are operational, or E and F are operational. What is the probability the apparatus works?
2. Consider a different scenario, in which the elements A and C, D and F and B and E are pairwise linked; such that if either of them breaks down, then the linked element is not operational either. What is the probability that the apparatus works now?