Assignment9 – Propositional and First-Order Logic

Problem 9.1 (Calculi Comparison)

Prove (or disprove) the *validity* of the following *formulae* in i) *Natural Deduction* ii) *Tableau* and iii) *Resolution*:

1. $P \wedge Q \Rightarrow (P \vee Q)$
2. $(A \vee B) \wedge (A \Rightarrow C) \wedge (B \Rightarrow C) \Rightarrow C$
3. $((P \Rightarrow Q) \Rightarrow P) \Rightarrow P$
4. Can you identify any advantages or disadvantage of the *calculi*, and in which situations?

Problem 9.2 (Equivalence of CSP and SAT)

We consider

- *CSPs* $\langle V, D, C\rangle$ with *finite domains* as before
- *SAT problems* $\langle V, A\rangle$ where V is a set of propositional variables and A is a propositional formula over V.

We will show that these problem classes are equivalent by reducing their instances to each other.

1. Given a SAT instance $P = \langle V, A\rangle$, define a CSP instance $P' = \langle V', D', C'\rangle$ and two *bijections*:
   - f mapping satisfying assignments of P to solutions of P',
   - and f' the inverse of f.

   We already know that *constraint networks* are equivalent to *higher-order CSPs*. Therefore, it is sufficient to give a *higher-order CSP*.
2. Given a CSP instance $\langle V, D, C\rangle$, define a SAT instance (V', A') and *bijections* as above.

Problem 9.3 (Induction)

Use structural induction on terms and formulas to define a function C that maps every term/formula to the number of *free variable occurrences*. For example, $C(\forall x.P(x, x, y, y, z)) = 3$ because the argument has 2 *free occurrences* of y and one of z.

Hint: Use an auxiliary function $C'(V, A)$ that takes the set V of bound variables and a term/formula A. Define C' by structural induction on A. Then define $C(A) = C'(\emptyset, A)$.

Problem 9.4 (First-Order Semantics)

Let $= \in \Sigma_2^p$, $P \in \Sigma_1^p$ and $+ \in \Sigma_2^f$. We use the semantics of first-order logic without equality.

Prove or refute the following formulas semantically. That means you must show that $I_\varphi(A) = T$ for all models I and assignments φ (without using a proof calculus) or to give some I, φ such that $I_\varphi(A) = F$.

1. $P(X)$
2. $\forall X.\forall Y. = (+(X,Y), +(Y,X))$
3. $\exists X.P(X) \Rightarrow (\forall Y.P(Y))$
4. $P(Y) \Rightarrow (\exists X.P(X))$