Assignment8 – Calculi for Propositional Logic

Problem 8.1 (FOL-Signatures)

1. Model the following situation as a FOL signature. (FOL and PLNQ signatures are the same.)
   - We have constants (= nullary functions) called `zero` and `one`.
   - We have a binary function called `plus`.
   - We have a unary function called `minus`.
   - We have a binary predicate called `less`.
2. Now consider the signature given by
   - $\Sigma_0^f = \{a, b\}$
   - $\Sigma_1^f = \{f, g\}$
   - $\Sigma_2^f = \{h\}$
   - $\Sigma_0^p = \{p\}$
   - $\Sigma_1^p = \{q\}$
   - $\Sigma_2^p = \{r\}$
   - all other sets empty
3. Give a term over this signature that uses all function symbols

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Solution: $\Sigma_0^f = \{\texttt{zero}, \texttt{one}\}$, $\Sigma_1^f = \{\texttt{minus}\}$, $\Sigma_2^f = \{\texttt{plus}\}$, $\Sigma_2^p = \{\texttt{less}\}$, and all other sets are empty

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4. Give a formula over this signature that uses all function and predicate symbols

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Solution: E.g., $t = h(f(a), g(b))$ for the term $r(t \wedge t) \wedge q(t) \wedge p$ for the formula

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Problem 8.2 (Natural Deduction)

Prove the following formula using the propositional Natural Deduction calculus.

$$(A \vee B) \wedge (A \Rightarrow C) \wedge (B \Rightarrow C) \Rightarrow C$$

Solution:

(1)	1	$(A \vee B) \wedge (A \Rightarrow C) \wedge (B \Rightarrow C)$	Assumption
(2)	1	$A \vee B$	$\mathcal{ND}_0 \wedge E_l$ (on 1)
(3)	1	$(A \Rightarrow C) \wedge (B \Rightarrow C)$	$\mathcal{ND}_0 \wedge E_r$ (on 1)
(4)	1	$A \Rightarrow C$	$\mathcal{ND}_0 \wedge E_l$ (on 3)
(5)	1	$B \Rightarrow C$	$\mathcal{ND}_0 \wedge E_r$ (on 3)
(6)	1,6	A	Assumption
(7)	1,6	C	$\mathcal{ND}_0 \Rightarrow E$ (on 4 and 6)
(8)	1,8	B	Assumption
(9)	1,8	C	$\mathcal{ND}_0 \Rightarrow E$ (on 5 and 8)
(10)	1	C	$\mathcal{ND}_0 \vee E$ (on 2, 7 and 9)
(11)		$(A \vee B) \wedge (A \Rightarrow C) \wedge (B \Rightarrow C) \Rightarrow C$	$\Rightarrow I$ (on 1 and 10)

Problem 8.3 (Proving in Tableau Calculus)

We use the *propositional variables* P, Q, and R and define *formulae* A, B, and C by

$$A = Q \wedge (Q \Rightarrow R)$$

$$B = P \Rightarrow A$$

$$C = P \Rightarrow R$$

Prove the *formula* $B \Rightarrow C$ using the *propositional tableau calculus* $\mathcal{T}_0$.

Solution:

$$
\begin{array}{c}
(B \Rightarrow C)^{\mathsf{F}} \\ \hline
B^{\mathsf{T}} \\
C^{\mathsf{F}} \\
P^{\mathsf{T}} \\
R^{\mathsf{F}} \\
\begin{array}{c|c}
P^{\mathsf{F}} & A^{\mathsf{T}} \\
\text{close on } P & Q^{\mathsf{T}} \\
 & (Q \Rightarrow R)^{\mathsf{T}} \\
 & \begin{array}{c|c} Q^{\mathsf{F}} & R^{\mathsf{T}} \\ \text{close on } Q & \text{close on } R \end{array}
\end{array}
\end{array}
$$

$\perp$ for closing is acceptable.

Problem 8.4 (Logical Systems)

Fix a set V of propositional variables. We define a logical system $\langle L, K, \models \rangle$. (Note: This logical system is different from the ones in the lecture and only used here as an exercise.)

- L is the powerset of V, i.e., a formula is a set of propositional variables.
- K is the set of functions $V \to \{F, T\}$.
- For $A \in L$ and $M \in K$, $M \vDash A$ holds if $M(p) = T$ for all $p \in A$.

1. Give examples of formulas that are
    1. satisfiable
    2. falsifiable
    3. unsatisfiable
    4. valid

   Give a sound and complete calculus for this logical system.

   Solution: Assume some $p \in V$.

    1. $\{p\}$ (satisfied if $K(p) = T$)
    2. $\{p\}$ (falsified if $K(p) = F$)
    3. No such formula exists
    4. $\emptyset$ is the only valid formula

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2. Consider the relation $H \vdash A$ holding if $A = \bigcup_{h \in H} h$. Check if $\vdash$ is a derivation relation.

   Solution: It is not (unless $|V| \leq 1$). For example, put $H = \{\{p\}, \{q\}\}$ and $A = \{p\}$. Then $\bigcup_{h \in H} h = \{p, q\}$ and thus $H \nvdash A$ even though $A \in H$.
   It becomes a derivation relation if we use $\subseteq$ instead of $=$ in the definition.
