Short Proofs in Strong Proof Systems

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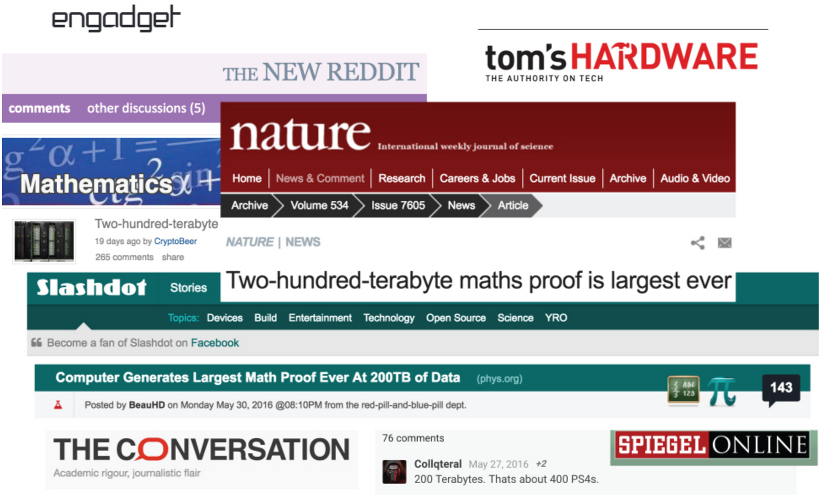
"The Largest Math Proof Ever"



Proofs of Unsatisfiability

Beyond Resolution

Finding Short Proofs

Short Proofs via Preprocessing

Future Work and Challenges

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Certifying Satisfiability and Unsatisfiability

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  - Just consider a satisfying assignment: $x\overline{y}z$

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- Certifying unsatisfiability is not so easy:
  - If a formula has n variables, there are 2^n possible assignments.
  - ➥ Checking whether every assignment falsifies the formula is costly.
  - More compact certificates of unsatisfiability are desirable.
    - ➥ Proofs

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  - Proofs are efficiently (usually polynomial-time) checkable...
    ... but can be of exponential size with respect to a formula.

- Example: Resolution proofs
  - A resolution proof is a sequence $C_1, \ldots, C_m$ of clauses.
  - Every clause is either contained in the formula or derived from two earlier clauses via the resolution rule:

$$\frac{C \vee x \qquad \bar{x} \vee D}{C \vee D}$$

  - C_m is the empty clause (containing no literals), denoted by $\perp$.
  - There exists a resolution proof for every unsatisfiable formula.

Resolution Proofs

- Example: $F = (\overline{x} \vee \overline{y} \vee z) \wedge (\overline{z}) \wedge (x \vee \overline{y}) \wedge (\overline{u} \vee y) \wedge (u)$
- Resolution proof:
  $(\overline{x} \vee \overline{y} \vee z), (\overline{z}), (\overline{x} \vee \overline{y}), (x \vee \overline{y}), (\overline{y}), (\overline{u} \vee y), (\overline{u}), (u), \perp$

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$$\dfrac{\dfrac{\bar{u} \lor y \qquad \dfrac{\dfrac{\bar{x} \lor \bar{y} \lor z \qquad \bar{z}}{\bar{x} \lor \bar{y}} \qquad x \lor \bar{y}}{\bar{y}}}{\bar{u}} \qquad u}{\bot}$$

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- Drawbacks of resolution:
  - For many seemingly simple formulas, there are only resolution proofs of exponential size.
  - State-of-the-art solving techniques are not succinctly expressible.

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Reduce the size of the proof by only storing added clauses

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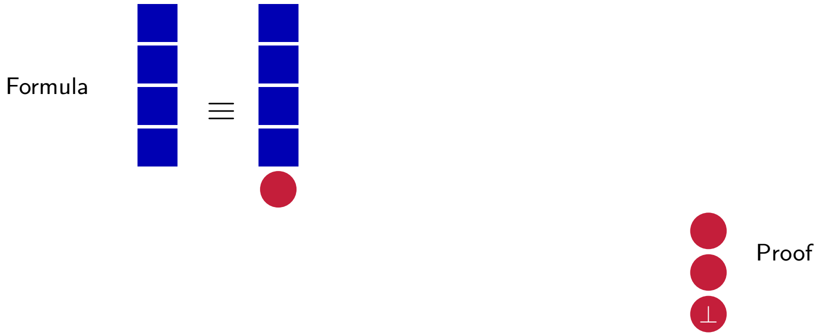

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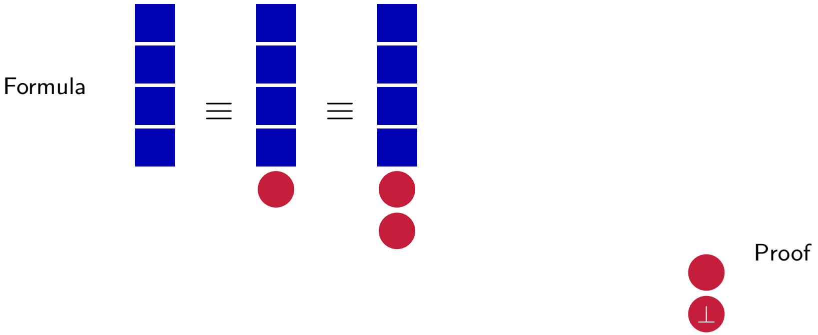

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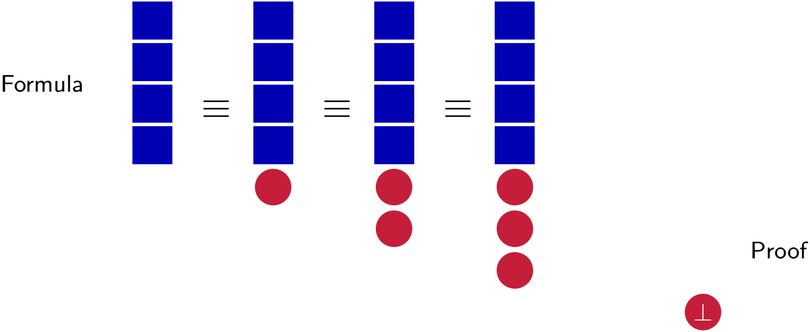

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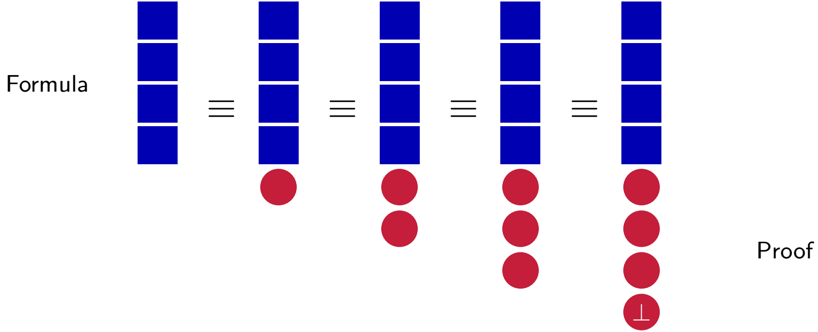

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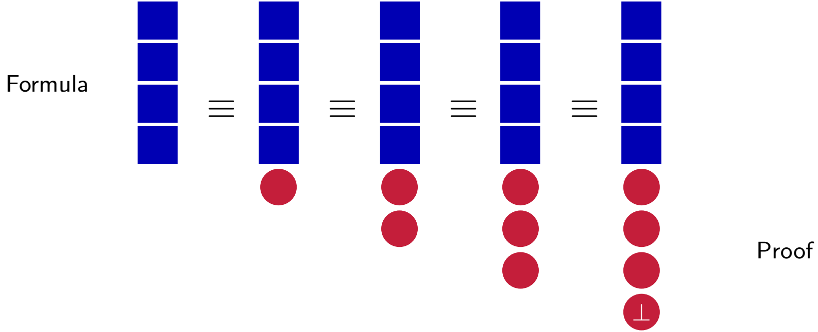

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- Checking redundancy should be efficient.

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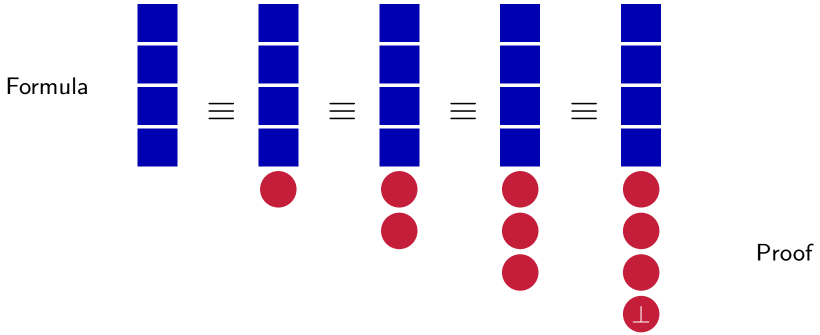

- Clauses whose addition preserves satisfiability are redundant.
- Checking redundancy should be efficient.

➥ Idea: Only add clauses that fulfill an efficiently checkable redundancy criterion.

Reverse Unit Propagation

- Unit propagation (UP) satisfies unit clauses by assigning their literal to true (until fixpoint or a conflict).
- Let F be a formula, C a clause, and α the smallest assignment that falsifies C. C is implied by F via UP (denoted by $F \vdash_1 C$) if UP on $F|_\alpha$ results in a conflict.

Example

$$\begin{aligned} F = \ & (a \lor b \lor \bar{c}) \land (\bar{a} \lor \bar{b} \lor c) \land (b \lor c \lor \bar{d}) \land (\bar{b} \lor \bar{c} \lor d) \ \land \\ & (a \lor c \lor d) \land (\bar{a} \lor \bar{c} \lor \bar{d}) \land (\bar{a} \lor b \lor d) \land (a \lor \bar{b} \lor \bar{d}) \end{aligned}$$

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$$\dfrac{\dfrac{(a \lor c \lor d) \qquad (b \lor c \lor \bar{d})}{(a \lor b \lor c)} \qquad (a \lor b \lor \bar{c})}{(a \lor b)}$$

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Future Work and Challenges

Traditional Proofs vs. Interference-Based Proofs

- In traditional proof systems, everything that is inferred, is logically implied by the premises.

$$\frac{C \lor x \qquad \overline{x} \lor D}{C \lor D}\ \text{(RES)} \qquad\qquad \frac{A \qquad A \rightarrow B}{B}\ \text{(MP)}$$

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➥ Inference rules reason about the presence of facts.

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➥ Inference rules reason about the presence of facts.

- If certain premises are present, infer the conclusion.

- Different approach: Allow not only implied conclusions.
  - Require only that the addition of facts preserves satisfiability.
  - Reason also about the absence of facts.

➥ This leads to interference-based proof systems.

Early work on reasoning beyond resolution

The early SAT decision procedures used the Pure Literal rule [Davis and Putnam 1960; Davis, Logemann and Loveland 1962]:

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Extended Resolution (ER) [Tseitin 1966]

- Combines resolution with the Extension rule:

$$\frac{x \notin F \qquad \overline{x} \notin F}{(x \vee \overline{a} \vee \overline{b}) \wedge (\overline{x} \vee a) \wedge (\overline{x} \vee b)} \text{ (ER)}$$

- Equivalently, adds the definition $x := \mathrm{AND}(a, b)$
- Can be considered the first interference-based proof system
- Is very powerful: No known lower bounds

Short Proofs of Pigeon Hole Formulas [Cook 1967]

Can $n+1$ pigeons be in n holes (at-most-one pigeon per hole)?

Resolution proofs are exponential in n [Haken 1985]

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However, these proofs require introducing new variables:

- Hard to find such proofs automatically
- Existing ER approaches produce exponentially large proofs
- How to get rid of this hurdle? First approach: blocked clauses...

Blocked Clauses [Kullmann 1999]

Definition (Block Clause)

A clause $(C \lor x)$ is a blocked on x w.r.t. a CNF formula F if for every clause $(D \lor \overline{x}) \in F$, resolvent $C \lor D$ is a tautology.

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Example

Consider the formula $(a \vee b) \wedge (a \vee \overline{b} \vee \overline{c}) \wedge (\overline{a} \vee c)$.
First clause is not blocked.
Second clause is blocked by both a *and* $\overline{c}$.
Third clause is blocked by c.

Theorem

Adding or removing a blocked clause preserves (un)satisfiability.

Blocked Clause Addition and Blocked Clause Elimination

The Blocked Clause proof system (BC) combines the resolution rule with the addition of blocked clauses.

- BC generalizes ER [Kullmann 1999]
- Recall

$$\frac{x \notin F \qquad \overline{x} \notin F}{(x \lor \overline{a} \lor \overline{b}) \land (\overline{x} \lor a) \land (\overline{x} \lor b)} \text{(ER)}$$

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Blocked clause elimination used in preprocessing and inprocessing

- Simulates many circuit optimization techniques
- Removes redundant Pythagorean Triples

DRAT: An Interference-Based Proof System

- DRAT is a popular interference-based proof system
- DRAT allows adding RATs (defined below) to a formula.
  - It can be efficiently checked if a clause is a RAT.
  - RATs are not necessarily implied by the formula.
  - But RATs are redundant: their addition preserves satisfiability.
- DRAT also allows clause deletion
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Definition (Resolution Asymmetric Tautology)

A clause $(C \lor x)$ is a resolution asymmetric tautology (RAT) on x w.r.t. a CNF formula F if for every clause $(D \lor \overline{x}) \in F$, $C \lor D$ is implied by F via unit-propagation, i.e., $F \vdash_1 C \lor D$.

Proof Search in Strong Proof Systems

Existence of Short Proofs

Extended Resolution '70

↓

Frege Systems

↓

Cutting Plane Method '62

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Resolution '60 / CDCL '97

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Regular Resolution

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Tree Resolution / DPLL '62

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Analytic Tableaux '68

logical equivalence

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logical equivalence

Finding Short Proofs

satisfiability equivalence

Propagation Red. (PR) '17

Set Prop. Red. (SPR) / SDCL '17

Res. Asym. Taut. (RAT) '12

Blocked Clauses (BC) '99

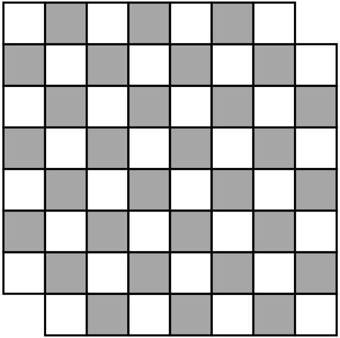
Extended Resolution (ER) '70

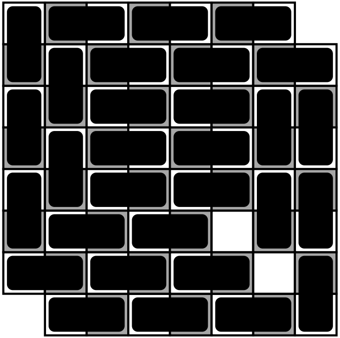
Express solving techniques compactly [Järvisalo, Heule, and Biere '12]

Short proofs without new variables [Heule, Kiesl, and Biere '17]

Mutilated Chessboards: “A Tough Nut to Crack” [McCarthy]

Can a chessboard be fully covered with dominos after removing two diagonally opposite corner squares?





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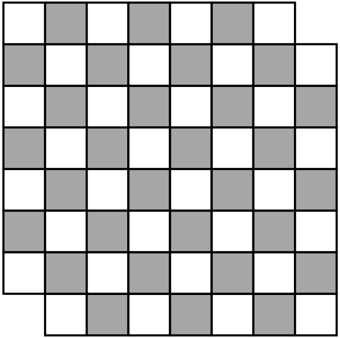
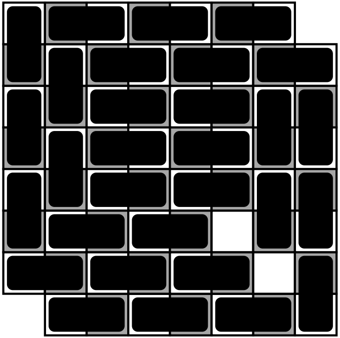


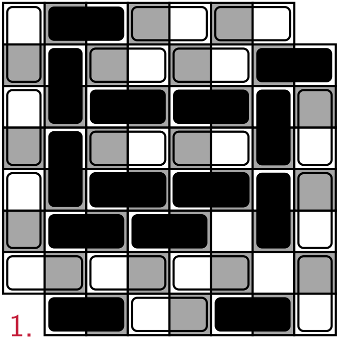


Easy to refute based on the following two observations:

- There are more white squares than black squares; and
- A domino covers exactly one white and one black square.

Without Loss of Satisfaction

One of the crucial techniques in SAT solvers is to generalize a conflicting state and use it to constrain the problem.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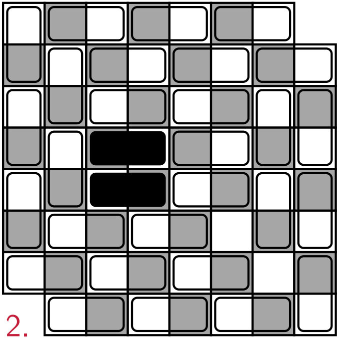
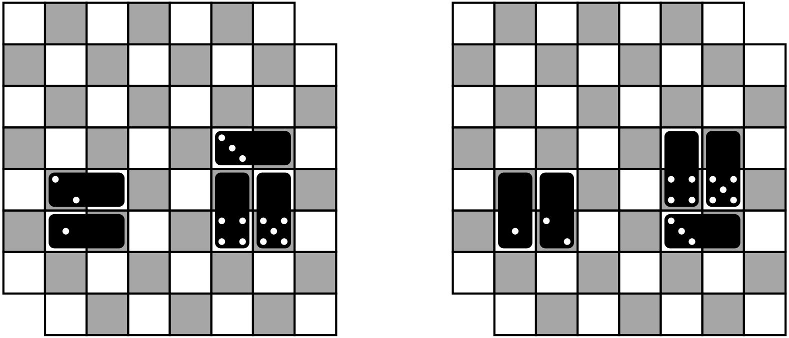

2.

The used proof system can have a big impact on the size:

1. Resolution can only reduce the 30 dominos to 14 (left); and
2. "Without loss of satisfaction" can reduce them to 2 (right).

Mutilated Chessboards: An alternative proof

Satisfaction-Driven Clause Learning (SDCL) is a new solving paradigm that finds proofs in the PR proof system [HKB '17]



SDCL can detect that the above two patterns can be blocked

- This reduces the number of explored states exponentially
- We produced SPR proofs that are linear in the formula size

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Autarkies

A non-empty assignment α is an autarky for formula F if every clause $C \in F$ that is touched by α is also satisfied by α.

A pure literal and a satisfying assignment are autarkies.

Example

Consider the formula $F := (x \vee y) \wedge (x \vee \overline{y}) \wedge (\overline{y} \vee \overline{z})$.

- $\alpha_1 = \overline{z}$ is an autarky: $(x \vee y) \wedge (x \vee \overline{y}) \wedge (\overline{y} \vee \overline{z})$.
- $\alpha_2 = x\,\overline{y}\,z$ is an autarky: $(x \vee y) \wedge (x \vee \overline{y}) \wedge (\overline{y} \vee \overline{z})$.

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Given an assignment α, $F|_\alpha$ denotes a formula F without the clauses satisfied by α and without the literals falsified by α.

Theorem ([Monien and Speckenmeyer 1985])

Let α be an autarky for formula F.
Then, F and $F|_\alpha$ are satisfiability equivalent.

Conditional Autarkies

An assignment $\alpha = \alpha_{\text{con}} \cup \alpha_{\text{aut}}$ is a conditional autarky for formula F if α_{aut} is a non-empty autarky for $F|\alpha_{\text{con}}$.

Example

Consider the formula $F := (x \vee y) \wedge (x \vee \overline{y}) \wedge (\overline{y} \vee \overline{z})$.
Let $\alpha_{\text{con}} = x$ and $\alpha_{\text{aut}} = \overline{y}$, then $\alpha = \alpha_{\text{con}} \cup \alpha_{\text{aut}} = x\,\overline{y}$ is a conditional autarky for F:

$$\alpha_{\text{aut}} = \overline{y} \text{ is an autarky for } F|\alpha_{\text{con}} = (\overline{y} \vee \overline{z}).$$

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Let $\alpha = \alpha_{\text{con}} \cup \alpha_{\text{aut}}$ be a conditional autarky for formula F.
Then F and $F \wedge (\alpha_{\text{con}} \rightarrow \alpha_{\text{aut}})$ are satisfiability-equivalent.

In the above example, we could therefore learn $(\overline{x} \vee \overline{y})$.

Finding Conditional Autarkies

The positive reduct of a formula F and an assignment α, denoted by $p(F, \alpha)$, is the formula that contains clause $C = \overline{\alpha}$ and all $assigned(D, \alpha)$ with $D \in F$ and D is satisfied by α.

Example

Consider the formula $F := (x \vee y) \wedge (x \vee \overline{y}) \wedge (\overline{y} \vee \overline{z})$.

- Let $\alpha_1 = x$, so $C_1 = (\overline{x})$ The positive reduct $p(F, \alpha_1) = (\overline{x}) \wedge (x) \wedge (x)$ is unsatisfiable.
- Let $\alpha_2 = x\,y$, so $C_2 = (\overline{x} \vee \overline{y})$. The positive reduct $p(F, \alpha_2) = (\overline{x} \vee \overline{y}) \wedge (x \vee y) \wedge (x \vee \overline{y})$ is satisfiable.

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The positive reduct of a formula F and an assignment α, denoted by $p(F, \alpha)$, is the formula that contains clause $C = \overline{\alpha}$ and all $assigned(D, \alpha)$ with $D \in F$ and D is satisfied by α.

Example

Consider the formula $F := (x \vee y) \wedge (x \vee \overline{y}) \wedge (\overline{y} \vee \overline{z})$.

- Let $\alpha_1 = x$, so $C_1 = (\overline{x})$ The positive reduct $p(F, \alpha_1) = (\overline{x}) \wedge (x) \wedge (x)$ is unsatisfiable.
- Let $\alpha_2 = x\,y$, so $C_2 = (\overline{x} \vee \overline{y})$. The positive reduct $p(F, \alpha_2) = (\overline{x} \vee \overline{y}) \wedge (x \vee y) \wedge (x \vee \overline{y})$ is satisfiable.

Theorem ([Heule, Kiesl, Biere '17B])

Given a formula F and an assignment α. Every satisfying assignment ω of $p(F, \alpha)$ is a conditional autarky of F.

Satisfaction-Driven Clause Learning [Heule, Kiesl, Biere '17B]

SDCL generalizes CDCL and finds proofs in the SPR proof system.

CDCL in a nutshell:

1. Main loop combines efficient problem simplification with cheap, but effective decision heuristics; ($> 90\%$ of time)
2. Reasoning kicks in if the current state is conflicting;
3. The current state is analyzed and turned into a constraint;
4. The constraint is added to the problem, the heuristics are updated, and the algorithm (partially) restarts.

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Short proofs for problems that are hard for resolution

including pigeonhole, Tseitin, and mutilated chessboard problems

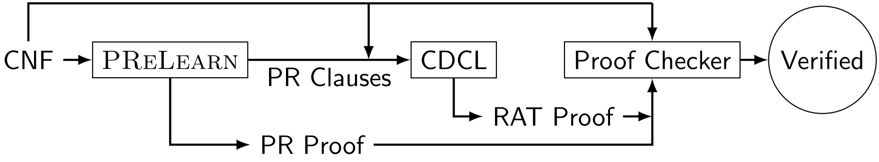
Preprocessing: PReLearn [Reeves, Heule, Bryant 2022]

Key observation: If PR clauses are useful, then we only need PR clauses of length at most 2.

Idea: only look for PR clauses of length at most 2

For each literal $l \in F$, check if $(l \vee k)$ is a PR clause.

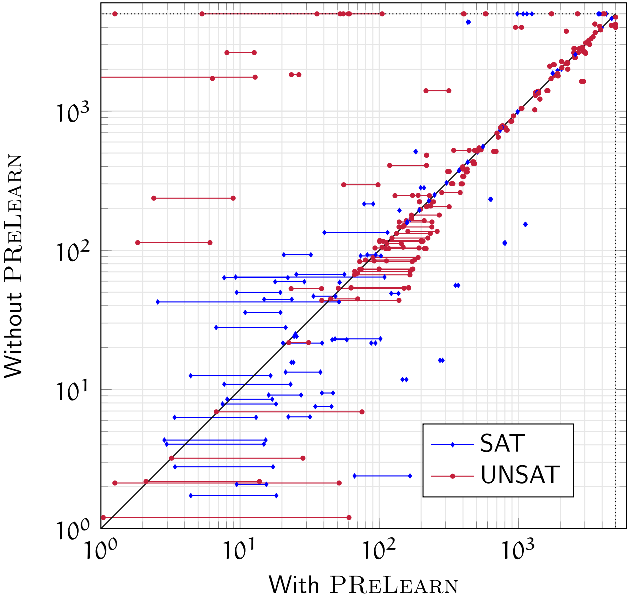
- Restrict the search by k occurring in $F \setminus F|l$.
- Use the positive reduct to check for PR.



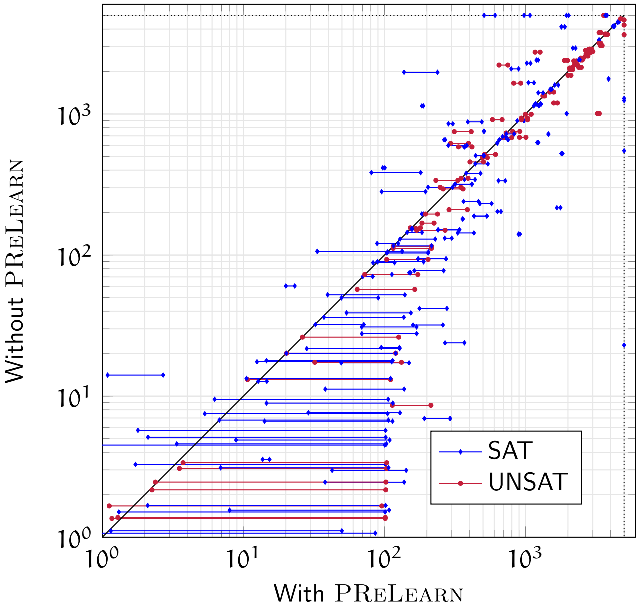
Preprocessing: Mutilated Chessboard

N	Time	Rounds	Avg.	Units	Bin.	Avg. Units	Avg. Bin.
8	0.14	1	0.14	30	164	30.00	164.00
12	4.94	1	4.94	103	1,045	103.00	1,045.00
16	62.47	2	31.23	195	3,988	97.50	1,994.00
20	513.12	6	85.52	339	1,4470	56.50	2,411.67
24	4,941.38	26	190.05	512	64,038	19.69	2,463.00

Preprocessing: Small Formulas (up to 10K clauses)



Preprocessing: Medium-Sized Formulas (10K-50K clauses)



Preprocessing: Biggest Impact on Performance

Size	Value	With	Without	Clauses	Formula	Year
0-10k	UNSAT	1.26	–	2,033	ph12*	2013
0-10k	UNSAT	35.69	–	20,179	Pb-chnl15-16_c18*	2019
0-10k	UNSAT	105.01	–	46,759	Pb-chnl20-21_c18	2019
0-10k	UNSAT	59.99	–	1,633	randomG-Mix-n17-d05	2021
0-10k	UNSAT	61.08	–	1,472	randomG-n17-d05	2021
0-10k	UNSAT	407.51	–	1,640	randomG-n18-d05	2021
0-10k	UNSAT	584.95	–	1,706	randomG-Mix-n18-d05	2021
0-10k	SAT	1,082.62	–	9,650	fsf-300-354-2-2-3-2.23.opt	2013
0-10k	SAT	1,250.82	–	10,058	fsf-300-354-2-2-3-2.46.opt	2013
10k-50k	SAT	1,076.34	–	804	sp5-26-19-bin-stri-flat	2021
10k-50k	SAT	608.48	–	901	sp5-26-19-una-nons-tree	2021
10k-50k	SAT	–	22.99	254	Ptn-7824-b13	2016
10k-50k	SAT	–	549.27	133	Ptn-7824-b09	2016
10k-50k	SAT	–	1,246.42	39	Ptn-7824-b02	2016
10k-50k	SAT	–	1,290.49	121	Ptn-7824-b08	2016
10k-50k	UNSAT	–	3,650.21	31,860	rphp4_110_shuffled	2016
10k-50k	UNSAT	–	4,273.88	31,531	rphp4_115_shuffled	2016

Future Work: Arbitrarily Complex Solvers

Verifying efficient automated reasoning tools is a daunting task:

- Tools are constantly modified and improved; and
- Even top-pier and "experimentally correct" solvers turned out to be buggy. [Järvisalo, Heule, Biere '12]

Verified checkers of certificates in strong proof systems:

- Don't worry about correctness or completeness of tools;
- Facilitates making tools more complex and efficient; while
- Full confidence in results. [Heule, Hunt, Kaufmann, Wetzler '17]

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ACL2 Isabelle CAKEML

Formally verified checkers now also used in industry

Theoretical Challenges

Lower bounds for interference-based proof systems with new variables will be hard, but what about without new variables?

- Lower bound for BC w/o new variables? Pigeon-hole formulas?
- Lower bound for SET w/o new variables? Tseitin formulas?
- Lower bound for PR w/o new variables?!

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Can we design stronger proof systems that make it even easier to compute short proofs?

Practical Challenges

The current version of SDCL is just the beginning:

- Which heuristics allow learning short PR clauses?
- How to construct an AnalyzeWitness procedure?
- Can the positive reduct be improved?

Can local search be used to find short proofs of unsatisfiability?

Constructing positive reducts (or similar formulas) efficiently:

- Generating a positive reduct is more costly than solving them
- Can we design data-structures to cheaply compute them?