On Finite Convergence of the Modal Mu-Calculus [1]

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"Finite convergence is guaranteed if and only if bisim. quotient of a structure is finite."

[1] Joint work with Florian Bruse and Martin Lange, University of Kassel, Germany.

Understanding (Finite) Convergence of Mu-Formulas

Mu-Formula: $\varphi := \text{lfp } X.\, \text{here}(aabaa) \text{ or next}(X)$
expresses: 'does *aabaa* eventually hold'

Iterative Approximation:

$$\varphi^1 := \text{here}(aabaa) \text{ or next}(\bot) \qquad \varphi^2 := \text{here}(aabaa) \text{ or next}(\varphi^1)$$

$$\varphi^3 := \text{here}(aabaa) \text{ or next}(\varphi^2) \qquad \cdots$$

$$\text{converges: } \varphi^{i+1} \equiv \varphi^i \quad (\Rightarrow \varphi^i \equiv \varphi)$$

Infinite Word:

$$\overleftarrow{a\;a}\; a\; b\; \overleftarrow{a\;a\;a}\; a\; b\; \overleftarrow{a\;a\;a}\; a\; b\; a\; a\; b\; b\; a\; \overleftarrow{b\;b\;a}\; a\; b\; \overleftarrow{a\;a\;a}\; a\; b\; a\; a\; \cdots$$

Back to our Initial Question

"Finite convergence is guaranteed if and only if bisim. quotient of a structure is finite."

Two directions:

- "$\Leftarrow$" If a structure has a finite bisimulation quotient then finite convergence of all mu-formulas is guaranteed.
- "$\Rightarrow$" If all mu-formulas have finite convergence over some structure then the structure has a finite bisimulation quotient.

A Suitable, Infinite Word w

$$a\ a\ a\ b\ a\ a\ a\ a\ b\ a\ a\ a\ a\ b\ a\ a\ b\ b\ a\ b\ b\ a\ a\ b\ a\ a\ a\ a\ b\ a\ a\ \cdots$$

$$w = \alpha_0\,\alpha_1\,\alpha_2\,\cdots$$

$$\alpha_0 := a \qquad \alpha_{i+1} := \alpha_i^2\beta_i\alpha_i^2$$
$$\beta_0 := b \qquad \beta_{i+1} := \beta_i^2\alpha_i\beta_i^2$$

How the word w guarantees finite convergence

Mu-Calculus

lfp-Operator

"**eventually** some pattern"

gfp-Operator

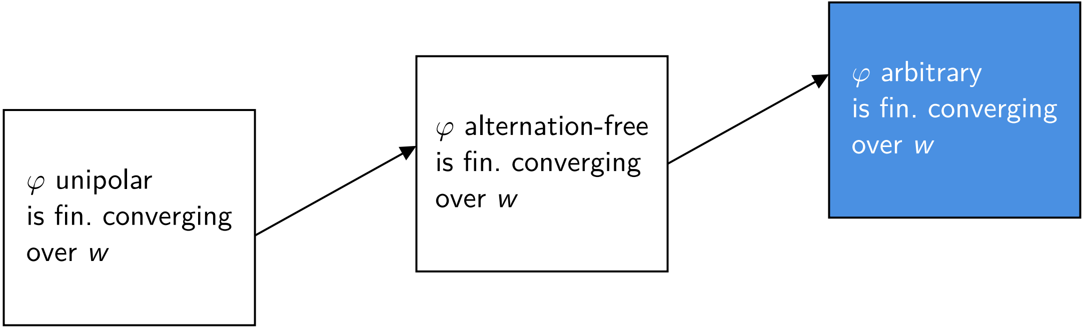
"**always** some pattern"

generally, more complicated...

Word

"there are **infinitely many patterns** and each occurs at a **bounded distance.**"

Proving Finite Convergence of all Mu-Formulas over w



Implications of our Findings

- Infinite bisimulation does **not** imply infinite convergence! (mu-calculus)

- And this is due to a lack of expressive power in the Mu-Calculus:

  HFL-fomula: $(\text{gfp } X.\, \text{map } \alpha_i, \beta_i.\, (\alpha_{i+1} \top) \wedge (X\, \alpha_{i+1}\, \beta_{i+1}))\, \alpha_0\, \beta_0$
  expresses: "all α_i start here"

Further research:

- How far can we stretch the pattern of this word? General structures?
- Further questions regarding finite convergence of HFL formulas.